Power Series as Functions

Idea: can view a power series

\[ \sum c_n (x-a)^n \]

as a function \( f(x) \) on its
interval of convergence.

Amazingly: many well-known functions can be represented as power series
we already know are e.g.

if \( |x| < 1 \) then \( f(x) = \frac{1}{1-x} \)

is equal to \( \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots \)

- of course if \( |x| \geq 1 \), \( \sum x^n \) diverges
and so \( \frac{1}{1-x} \) no longer valid

- but we'll see: representing functions \( f(x) \)
as power series on even parts of their
domain can be useful!
(2) \textbf{Ex:} express } f(x) = \frac{1}{1+x^2} \text{ as a power series and find the interval of convergence.}

\text{Sol'n: we know } \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \quad \text{as long as } |t|<1.

\text{So let } t = -x^2 \text{ we get}

\frac{1}{1-(-x^2)} = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n

= \sum_{n=0}^{\infty} (-1)^n x^{2n}

= 1 - x^2 + x^4 - x^6 + \ldots

\text{as long as } |x^2| < 1, \text{ i.e. } x^2 < 1, \text{ i.e. } |x| < 1

\text{because } \sum_{n=0}^{\infty} (-x^2)^n \text{ is a geometric series, (with } r = -x^2), \text{ thus it converges if and only if } 1 - x^2 < 1

\text{is the interval of convergence is } (-1,1).
Ex: Some $y$ for $f(y) = \frac{1}{x+2}$.

Soh: \[
\frac{1}{x+2} = \frac{1}{2+x} = \frac{1}{2} \left(1 + \frac{x}{2}\right)
\]
\[= \frac{1}{2} \left(1 - \left(-\frac{x}{2}\right)^n\right)
\]
\[= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n x^n
\]

which converges if $1 - \frac{x}{2} | < 1$

1. $1 - \frac{x}{2} | < 1$
2. $1 - \frac{x}{2} | > 1$
3. $-2 < x < 2$

So interval of convergence is $x$ core is $(-2, 2)$. 
Ex: Some \( p \) for \( f(x) = \frac{x^3}{x+2} \)

**Set \( p \):** We know we can distribute constants over sums: \( c \sum a_n = \sum c a_n \)

- We also know for a fixed \( x \) in \((-2, 2)\), the series \( \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^n n \) converges to \( \frac{1}{x+2} \).

- So... for \( x \) in \((-2, 2)\) we must have:

\[
\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n n x^n \quad \text{converges to} \quad \frac{x^3}{x+2}
\]

**De:** \( \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n n x^n \) is a power series representing \( \frac{x^3}{x+2} \) on interval \((-2, 2)\).
Recall: we think of a power series
\[ \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots \]
as an "infinite polynomial".

Polynomials are nice because they are easy to differentiate and integrate.

Turns out: same is true of power series!

**Theorem** If the power series \( \sum c_n(x-a)^n \) has radius of convergence \( R > 0 \), then the function \( f \) defined by:

\[ f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \]

is continuous and differentiable on \( (a-R, a+R) \) and further:

(i) \( f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \)

(ii) \( \int f(x) \, dx = \sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1} + C \)

\[ = \left( \frac{c_0(x-a)}{2} + \frac{c_1(x-a)^2}{3} + \frac{c_2(x-a)^3}{4} + \cdots \right) + C \]
and the radius of convergence for both series is also \( R \).

- Don't prove theorem (beyond scope)
- Note: this says: can differentiate and integrate power series term-by-term!

- Note: this also says: radius of convergence for differentiated and integrated series is same as original series, but what happens at the endpoints \( x = c + R \) and \( x = a - R \) may be different.

- Can use them to get power series rep's for new functions.

**Ex:** Find a power series representation for \( \frac{1}{1-x^2} \). What is the radius of convergence?

**Solt:** We know:

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots \quad \text{if} \quad |x| < 1
\]
\[
\frac{d}{dx} \left( \frac{1}{x^2} \right) = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \frac{d}{dx} (1 + x + x^2 + \ldots) \\
\text{by theorem} \\
\text{i.e.,} \quad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots \\
= \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n
\]

Thus also says: \( R = 1 \) for this series too. (Can check: \( \sum (n+1)x^n \) diverges if \( x = 1 \) or \(-1\).
De interval of convergence \( U (-1, 1) \).
In this case: same as original interval, but in general can be different.

**Ex:** Find power series rep'n for \( \frac{x}{(1-x)^2} \).

**Seth:** by above:
\[
\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + \ldots \quad \text{if} \left| x \right| < 1
\]

\[
\text{By} \quad \frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n = x (1 + 2x + 3x^2 + \ldots) \Rightarrow
\]
\[
= 2 + 2x^2 + 3x^3 + \ldots
\]

**Mult. by \( x \)**

Does not change radius
(4) \[ \exp: \text{Same as for } \ln(1+x). \]

**Proof:** a new trick.

Observe: \( \frac{d}{dx} \ln(1+x) = \frac{1}{1+x} \)

\[ = \frac{1}{1-(-x)} \]

\[ = \sum_{n=0}^\infty (-x)^n \]

\[ = \sum_{n=0}^\infty (-1)^n x^n \]

\[ = 1 - x + x^2 - x^3 + \ldots \quad \text{for } |x| < 1 \]

Hence \( \ln(1+x) = \sum_{n=0}^\infty (-1)^n x^n \)

\[ = \int (1 - x + x^2 - x^3 + \ldots) \]

\[ = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \ldots + C \]

\[ = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} x^n + C \quad \text{if } |x| < 1 \]

To solve for \( C \) with \( x = 0 \)

\[ \ln(1) = 0 + C \]

i.e. \( 0 = C \). Hence: \( \ln(1+x) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} x^n \)

(con't: check end point)
Ex: Some question for \( f(x) = \tan^{-1}(x) \)

\[ \frac{d}{dx} \tan^{-1}x = \frac{1}{1+x^2} = \frac{1}{1-(x^2)} \]

\[ = \sum_{n=0}^{\infty} (-x^2)^n \quad \text{if } |1-x^2|<1 \]
\[ \text{i.e. } |x|<1 \]

\[ = \sum_{n=0}^{\infty} (-1)^n x^{2n} \]

\[ = 1-x^2+x^4-x^6+... \]

So: \( \tan^{-1}x = \int 1-x^2+x^4-x^6+... \quad \text{for } |x|<1 \)

\[ = \left( x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + ... \right) + C \]

To find \( C \), let \( x = 0 \):

\( \tan^{-1}0 = 0 + C \)

\( \text{i.e. } 0 = C \)

So \( \tan^{-1}x = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + ... \)

\[ = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \checkmark \]
3x. Evaluate \( \int \frac{x}{1+x^3} \, dx \) as a power series. What is radius of convergence?

**Solution:** First we do \( \frac{x}{1+x^3} \)

We know:

\[
\frac{x}{1+x^3} = x \left( \frac{1}{1+x^3} \right) = x \left( \frac{1}{1-(-x^3)} \right)
\]

\[
= x \left( \sum_{n=0}^{\infty} (-x^3)^n \right) \quad \text{if} \quad |x^3| < 1
\]

\[
= x \left( \sum_{n=0}^{\infty} (-1)^n x^{3n} \right) = x \left( 1 - x^3 + x^6 - x^9 + \cdots \right)
\]

\[
= \sum_{n=0}^{\infty} (-1)^n x^{3n+1}
\]

\[
= x - x^4 + x^7 - x^{10} + \cdots
\]

For \( |x| < 1 \).

**Hence**

\[
\int \frac{x}{1+x^3} \, dx
\]

\[
= \int x - x^4 + x^7 - x^{10} + \cdots \, dx \quad \text{for} \quad |x| < 1
\]

\[
= \frac{1}{2} x^2 - \frac{1}{5} x^5 + \frac{1}{8} x^8 - \frac{1}{11} x^{11} + \cdots + C
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{3n+2} + C \quad \text{for} \quad |x| < 1
\]
Note: it turns out

\[ \int \frac{x}{1 + x^3} \, dx = \frac{1}{6} (-\ln(x^2 - x + 1) + 2\ln(2x+1) + 2\sqrt{3} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)) + C \]

... but this is hard to solve for directly.

Our power series representation of this integral is good enough for many applications.

Example: Approximate \( \int_0^{0.3} \frac{x}{1 + x^3} \, dx \) using the previous power series, to within two decimal places.

Solution: From above:

\[ \int \frac{x}{1 + x^3} = \frac{1}{2} x^2 - \frac{1}{5} x^5 + \frac{1}{8} x^8 - \cdots + C \]

\[ = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{3n+2} + C \text{ for all } x \]

Hence \( \int_0^{0.3} \frac{x}{1 + x^3} = \left( \frac{1}{2} (0.3)^2 - \frac{1}{5} (0.3)^5 + \frac{1}{8} (0.3)^8 - \cdots \right) \)

\[ - \left( \frac{1}{2} 0^2 - \frac{1}{5} 0^5 + \frac{1}{8} 0^8 - \cdots \right) \]

\[ = \frac{1}{2} (0.3)^2 - \frac{1}{5} (0.3)^5 + \frac{1}{8} (0.3)^8 - \cdots \]
This is a convergent alternating series. Let's approximate it only find three terms:

\[ a_0, a_1, a_2 \]

\[ \approx \frac{1}{2} (0.3)^2 - \frac{1}{3} (0.3)^3 + \frac{1}{8} (0.3)^4 \]

\[ = 0.049522208 \]

We know the error is bounded by next term:

\[ R_2 \leq |b_{n+1}| = \frac{1}{11} (0.3)^{11} = 1.61 \times 10^{-7} \]

i.e., \[ 0.0495 \ldots \] is within \[ 1.61 \times 10^{-7} \]

of \[ \int_0^2 \frac{x}{1+x^2} \, dx \ldots \] very better than the desired places!