Signed Measures.

Def $\mu : \Sigma \longrightarrow [-\infty,\infty]$ is a signed measure if ① $\mu(\phi)=0$, ② μ does not take on both $+\infty$ & $-\infty$ & ③ $A_i \in \Sigma$ disj $\Rightarrow$ $\mu(\bigcup_1^\infty A_i) = \sum_1^\infty \mu(A_i)$

Eg: μ_1, μ_2 two measures on X. Either $\mu_1(X)<\infty$ or $\mu_2(X)<\infty$. Then $\mu=\mu_1-\mu_2$ is a signed measure

Eg: (X,Σ,μ) a measure space. $f\in L^1(\mu)$. Define $\nu(A)=\int_A f\,d\mu$. Then ν is a signed measure (Dominated Convergence). [Note: If $f\geq 0$ then don't need $f\in L^1$ for ν to be a measure by monotone convergence (was on H.W.)]

Def: $A\in\Sigma$ is negative if $\mu(B)\leq 0$ $\forall B\subseteq A$, $B\in\Sigma$.

Prop: If $-\infty<\mu(A)<\infty$, $\exists B\subseteq A \ni$ B is negative & $\mu(B)\leq\mu(A)$

Pf: If $\mu(A)\geq 0$ then choose $B=\phi$.

Suppose $\mu(A)<0$. $\delta_1 = \sup\{\mu(E) \mid E\subseteq A,\ E\in\Sigma\}$. $\exists E_1\subseteq A \ni \mu(E_1) > \frac{\delta_1}{2}\wedge 1$

Note $\delta_1\geq 0$ ∵ $\phi\subseteq A$ & $\mu(\phi)=0$.

Note δ_1 could $=+\infty$. (this is why $\mu(E_1)>\frac{\delta_1}{2}\wedge 1$ & not just $>\frac{\delta_1}{2}$)

Let $\delta_{n+1} = \sup\{\mu(E) \mid E\subseteq A-\bigcup_1^n E_i\}$ $\exists E_{n+1}\subseteq A-\bigcup_1^n E_i \ni \mu(E_{n+1})>\frac{\delta_n}{2}\wedge 1$.

Let $B=A-\bigcup E_i$. Claim: $\mu(B)\leq\mu(A)$ & B is neg. (Claim $\Rightarrow$ Prop).

Pf of Claim: ① $\mu(\bigcup_1^\infty E_i)=\sum\mu(E_i)<\infty$. [Pf: $\mu(A)=\mu(B)+\sum\mu(E_i)$ & $\mu(A)<\infty \Rightarrow \sum\mu(E_i)<\infty$]

② $\Rightarrow (\delta_i)\longrightarrow 0$ (∵ $\sum \frac{\delta_n\wedge 1}{2}<\infty$)

③ $C\subseteq B \Rightarrow C\subseteq A-\bigcup_1^n E_i\ \forall n \Rightarrow \mu(C)\leq\delta_n\longrightarrow 0 \Rightarrow \mu(C)\leq 0$.
$\Rightarrow$ B is negative.

④ $\mu(B)=\mu(A)-\sum_1^\infty(\;)\leq\mu(A)$

QED.

Thm: (Hahn Decomposition) μ a signed measure on X. Then $X=P\cup N$, where P - positive & N - negative.

Remark: (Uniqueness) $X=P\cup N=P'\cup N' \Rightarrow P'-P,\ P-P',\ N-N',\ N'-N$ are all null sets.

(i.e. $A\subseteq(P'-P)\cup\cdots\cup(N'-N)\Rightarrow\mu(A)=0$). **Pf:** Exercise.

Pf of Thm: Without loss, $-\infty\notin\mu(\Sigma)$. $\alpha=\inf\{\mu(E) \mid E\in\Sigma\}$ ($\alpha\leq 0$, could be $-\infty$).

$\exists A_n \ni \alpha=\lim\mu(A_n)$. $\exists B_n\subseteq A_n$ negative, & $\mu(B_n)\leq\mu(A_n)$ (by prop).

Set $N=\bigcup B_n$. ① Clearly N negative.

② Say $C\subseteq N^c$, NTS $\mu(C)\geq 0$.

Note $C\subseteq N^c$, then $\alpha\leq\mu(B_n\cup C)\ \forall n \Rightarrow \alpha\leq\mu(B_n)+\mu(C)\longrightarrow\alpha+\mu(C)$

∴ $\mu(C)\geq 0$, provided we know $\alpha>-\infty$

Claim: $\alpha=\mu(N)$. ($\Rightarrow \alpha>-\infty \Rightarrow$ QED thm).

Pf of claim: $N\supseteq B_n$ & N negative $\Rightarrow \mu(N)\leq\mu(B_n)$

Also $\alpha\leq\mu(N)$ (Def of α).

∴ $\alpha\leq\mu(N)\leq\mu(B_n)\ \forall n \Rightarrow \alpha=\mu(N)>-\infty$.

QED.

Def: We say two +ve measures μ,ν are mutually singular if $\exists C\in\Sigma \ni$ $\forall A\in\Sigma$, $\mu(A\cap C)=0$ & $\nu(A\cap C^c)=0$. (Notation $\mu\perp\nu$).

Cor: (Jordan Decomposition) μ a signed measure on X. $\exists$ two +ve measures μ^+ & μ^- $\ni$ ① $\mu^+\perp\mu^-$ & ② $\mu=\mu^+-\mu^-$. This decomposition is unique.

Pf: $X=P\cup N$, be the Hahn Decomposition. Let $\mu^+(A)=\mu(A\cap P)$ & $\mu^-(A)=-\mu(A\cap N)$

Uniqueness: Say $\mu=\mu_1-\mu_2$ & $\mu_1\perp\mu_2$. $\Rightarrow \exists C\subseteq X \ni \forall A\in\Sigma$, $\mu_1(A\cap C)=0$ & $\mu_2(A\cap C^c)=0$. $\Rightarrow$ C is +ve (wrt μ) & C^c is −ve.

Uniqueness of Hahn $\Rightarrow$ $\mu_1=\mu^+$ & $\mu_2=\mu^-$.

Def: $|\mu|$ = variation of $\mu \overset{\text{def}}{=} \mu^+ - \mu^-$.

Def: $\|\mu\|$ = total variation of μ = $|\mu|(X)$.

Note: $|\mu(A)| \leq |\mu|(A)$. [Pf: $|\mu(A)| = |\mu_+(A) - \mu_-(A)| \leq \mu^+(A) + \mu^-(A)$

Def: $\mathcal{M} = \{$finite signed measures on $(X, \Sigma)\}$. Then $(\mathcal{M}, \|\ \|)$ is a Banach space.

Pf: ① $|\mu+\nu|(A) \leq |\mu|(A) + |\nu|(A)$: $(\mu+\nu)^+(A) = \sup_{B \subseteq A} \mu+\nu(B) \leq \underbrace{\sup \mu(B)}_{\mu^+(A)} + \underbrace{\sup \nu(B)}_{\nu^+(A)}$.

② Completeness $\to$ You check.

Rmk: $\|\mu\|$ not so useful: $\mu_n = \delta_{1/n}$. Would like $(\mu_n) \longrightarrow \delta_0$ in this norm.

But $\|\mu_n - \delta_0\| = 2\ \forall n$. $\Rightarrow (\mu_n) \not\longrightarrow \delta_0$

Absolute Continuity:

Def: $\nu \ll \mu$ if $\forall A \ni \mu(A) = 0$, $\nu(A) = 0$ (μ, ν +ve measures)

Eg: $\nu(A) = \int_A f\, d\mu$ $(f \geq 0)$. [Write $d\nu = f d\mu$, since $\int g\, d\nu = \int g f\, d\mu$.]

Thm: μ, ν σ-finite, +ve. & $\nu \ll \mu$. $\Rightarrow \exists! g \geq 0, g < \infty$ & $d\nu = g\, d\mu$.

Pf: Case I: μ, ν finite. $\mathcal{F} = \{ f \mid \int_A f\, d\mu \leq \nu(A)\ \forall A \in \Sigma \}$.

① $f, g \in \mathcal{F} \Rightarrow f \vee g \in \mathcal{F}$

② $f_n \in \mathcal{F}$ inc $\Rightarrow f = \lim f_n \in \mathcal{F}$ (Pf $\int_A f = \lim \int_A f_n \leq \nu(A)$.).

Choose $f_n \in \mathcal{F}$ & $\int_X f_n \to \sup_{f \in \mathcal{F}} \int_X f$.

replace f_n with $\max_{i \leq n} f_i \Rightarrow (f_n)$ inc $\Rightarrow f = \lim f_n \in \mathcal{F}$.

Let $\nu_0(A) = \nu(A) - \int_A f\, d\mu$. NTS $\nu_0 = 0$ (note $\nu_0 \geq 0$. & $\int_X f\, d\mu \leq \nu(X) < \infty \Rightarrow \overset{\text{a.e. } \mu.}{f < \infty}$).

Obs: $\int_A g\, d\mu \leq \nu_0(A)\ \forall A \Rightarrow f + g \in \mathcal{F} \Rightarrow \int_X g = 0 \Rightarrow g = 0$ a.e.

Claim $\forall \varepsilon > 0$, $\nu_0(A) \leq \varepsilon \mu(A)\ \forall A$. ($\Rightarrow \nu_0 = 0$).

Pf: $\nu_0 - \varepsilon\mu$ a signed meas. $X = P \cup N$.

$g = \varepsilon \chi_P$. $\int_A g\, d\mu = \varepsilon \mu(A \cap P) \leq \nu_0(A \cap P) \leq \nu_0(A)$

$\Rightarrow g = 0$ a.e. $\Rightarrow \mu(P) = 0 \Rightarrow \nu(P) = 0 \Rightarrow (\nu_0 - \varepsilon\mu)(P) = 0 \Rightarrow \nu_0 \leq \varepsilon\mu$.

QED.

② Uniq: $\int_A f = \int_A g = \nu(A)\ \forall A$. $f, g < \infty \Rightarrow \int_A (f - g) = 0\ \forall A$.

$\Rightarrow \int_{\{f > g\}} f - g = 0 \Rightarrow f \leq g$ a.e. III^{ly}, $f \geq g$ a.e. QED.

③ σ-finite: $X = \bigcup F_n$, $\nu(F_n) < \infty$, $\mu(F_n) < \infty$

$\nu_n = \nu(A \cap F_n)$, $\mu_n = \mu(A \cap F_n)$. $\nu_n = \int f_n\, d\mu_n$. $f_{n+1} = f_n$ on F_n.

Put $f = \lim f_n$. M.C. $\nu(A) = \lim \nu(A \cap F_n) = \lim \int_A f_n \chi_{F_n}\, d\mu \xrightarrow{\text{monotone}} \int_A f\, d\mu$ QED.