Measure Theory: Final.
Dec 12, 2014

- This is a closed book test. No calculators or computational aids are allowed.
- You have 3 hours. The exam has a total of 8 questions and 70 points.
- You may use without proof standard results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, CLEARLY state the result you are using.

Unless otherwise stated, we always assume the underlying measure space is \((X, \Sigma, \mu)\) and \(\mu\) is a positive measure. The Lebesgue measure on \(\mathbb{R}^d\) will be denoted by \(\lambda\).

10. Given \(f : X \to \mathbb{R}\) be measurable, and define \(F : \mathbb{R} \to [-\infty, \infty]\) by \(F(x) = \mu(f < x)\).

(a) True or false: If \(f\) is measurable, then \(F\) is left continuous. Prove it, or find a counter example.

(b) True or false: If \(f\) is measurable, then \(F\) is right continuous. Prove it, or find a counter example.

10. Let \(\mu\) be a finite measure on \(X\), and \(f : X \to \mathbb{R}\) be measurable. Decide whether the limits

\[
\lim_{n \to \infty} \int_X e^{-n|f|} \, d\mu \quad \text{and} \quad \lim_{n \to \infty} \int_X e^{+n|f|} \, d\mu
\]

necessarily exist. If yes, compute them. Prove your answer. [By convention, if a sequence approaches \(\infty\), we say the limit exists and is \(\infty\).]

10. Let \(E \subseteq \mathbb{R}^d\) be Lebesgue measurable. True or false:

   For any (possibly infinite) collection of balls \(\{B(x_\alpha, r_\alpha)\}_{\alpha \in A}\) such that

   \[
   \bigcup_{\alpha \in A} B(x_\alpha, r_\alpha) \supseteq E, \quad \text{and} \quad \sup_{\alpha \in A} r_\alpha < \infty,
   \]

   there exists a (possibly infinite) \(A' \subseteq A\) such that the sub-collection \(\{B(x_{\alpha'}, r_{\alpha'})\}_{\alpha' \in A'}\) is pairwise disjoint and

   \[
   \bigcup_{\alpha' \in A'} B(x_{\alpha'}, 5r_{\alpha'}) \supseteq E.
   \]

   Prove it, or find a counter example.

10. Let \(X\) be a compact metric space, \(C(X)\) denote the set of continuous real valued functions on \(X\). True or false:

   If \(\mu\) is a finite signed Borel measure on \(X\), then \(|\mu|| = \sup\left\{ \int_X f \, d\mu \mid f \in C(X) \text{ and } \sup |f| \leq 1 \right\}\).

   Prove it, or find a counter example. [You may not use the Riesz representation theorem for this question.]

10. If \(f, g \in L^2(\mathbb{R}^d)\) compute \((fg)^\wedge\) in terms of \(\mathcal{F}f\) and \(\mathcal{F}g\). Prove it. [Recall for \(f \in L^1(\mathbb{R}^d)\), we defined \(\hat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} \, dx\) to be the Fourier transform of \(f\), and \(\mathcal{F}\) denotes the extension of the Fourier transform to \(L^2\). Hint: First compute \((\hat{f} * \hat{g})^\wedge\) if \(f, g\) are Schwartz functions.]

6. Given \(f : \mathbb{R}^2 \to \mathbb{R}\), define \(G, H : \mathbb{R} \to \mathbb{R}\) by

\[
G(x) = \sup_{y \in \mathbb{R}} f(x, y) \quad \text{and} \quad H(x) = \begin{cases} \text{ess sup}_y f(x, y) & \text{if the function } y \mapsto f(x, y) \text{ is Lebesgue measurable,} \\ 0 & \text{otherwise.} \end{cases}
\]

Recall, ess sup is the essential supremum, defined by \(\text{ess sup}_y f(x, y) = \sup\{z \mid \lambda\{t \mid f(x, t) > z\} > 0\}\).

4. (a) True or false: If \(f\) is Lebesgue measurable, then so is \(G\). Prove it, or find a counter example.

4. (b) True or false: If \(f\) is Borel measurable, then so is \(G\). No proof required! Incorrect answers are worth no credit, blank answers half credit and correct answers full credit.
(c) True or false: If \( f \) is Borel measurable, then so is \( H \). Prove it, or find a counter example.

7. For any \( t \in [0, 1] \) and \( N \in \mathbb{N} \) define \( \Delta_{N,t}: \mathbb{R} \to \mathbb{R} \) by

\[
\Delta_{N,t}(x) = \sup \{ t + \frac{k}{N} \mid k \in \mathbb{Z} \text{ and } t + \frac{k}{N} \leq x \}.
\]

True or false:

If \( f \in L^1(\mathbb{R}) \) and \( \text{supp}(f) \subseteq [0, 1] \), then there exists an increasing sequence of integers \( N_k \to \infty \) such that \( \lim_{k \to \infty} \int_{\mathbb{R}} |f(x) - f(\Delta_{N_k,t}(x))| \, dx = 0 \) for almost every \( t \in [0, 1] \).

Prove it, or find a counter example. [Hint: Play with \( \int_0^1 \int_{\mathbb{R}} |f(x) - f(\Delta_{N,t}(x))| \, dx \, dt \).]

If you’ve completed the remainder of this exam and have time to spare, here is a fun question. This is for your entertainment only, and will not influence your grade.

8. Let \( X \) be a locally compact metric space, and \( \mu \) a regular Borel measure on \( X \). Suppose \( \mu(\{x\}) = 0 \) for every \( x \in X \). If \( F \in B(X) \) has finite measure, and \( 0 < \alpha < \mu(F) \), show that there exists \( A \in B(X) \) such that \( A \subset F \) and \( \mu(A) = \alpha \).