21-720 Measure Theory: Midterm.

October 11$^{\text{th}}$, 2013

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 80 mins. The exam has a total of 4 questions and 20 points.*
- *You may use any result from class or homework* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*
- *Good luck.*

In this exam, we always assume (X, Σ, μ) is a measure space. We use λ to denote the Lebesgue measure on $\mathbb{R}^d$.

$\boxed{5}$ 1. Let I be any (possibly uncountable) index set. For every $\alpha \in I$, let $f_\alpha : X \to \mathbb{R}$ be a measurable function. Let $f(x) = \sup\limits_{\alpha \in I} f_\alpha(x)$. Must f be measurable? Prove it or find a counter example.

2. Let $F : \mathbb{R}^d \to [0, \infty)$ be Lebesgue measurable. Consider the following statements:

   (i) $\displaystyle\int_{\mathbb{R}^d} F \, d\lambda < \infty$.

   (ii) Let (f_n) be a sequence of functions such that for all $n \in \mathbb{N}$, $f_n : \mathbb{R}^d \to \mathbb{R}$ is measurable, $|f_n| \leqslant F$, and the sequence (f_n) converges pointwise. Then $\lim\limits_{n\to\infty} \int_{\mathbb{R}^d} f_n \, d\lambda = \int_{\mathbb{R}^d} \lim\limits_{n\to\infty} f_n \, d\lambda$ for *every* such sequence (f_n).

$\boxed{1}$ (a) True or false: Statement (i) implies Statement (ii). (No proof required.)

$\boxed{4}$ (b) True or false: Statement (ii) implies Statement (i). Prove or provide a counter example.

$\boxed{5}$ 3. For $n \in \mathbb{N}$ let $A_n \in \Sigma$ be such that $\sum_1^\infty \mu(A_n) < \infty$. Define

$$A = \big\{ x \in X \bigm| \text{there exists infinitely many } k \in \mathbb{N} \text{ such that } x \in A_k \big\}.$$

Does $A \in \Sigma$? If yes, compute $\mu(A)$ in terms of the $\mu(A_n)$'s. Prove your answer.

$\boxed{5}$ 4. Let $f : \mathbb{R}^d \to [-\infty, \infty]$ be a Lebesgue measurable and integrable function such that $\int_I f \, d\lambda = 0$ for all cells I. Must $f = 0$ almost everywhere? Prove or find a counter example.