Diffusions

Def: A (time homogenious) diffusion is a process X that satisfies an SDE of the form

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t \quad \& \quad X_0 = x \text{ a.s.}$$ (Always assume b, σ are lipschitz & linear growth)

Notation: Let $X_t^{x,s} = X_t(x,s)$ be the solution of ⊛ with $X_s^{x,s} = x$ a.s. & $X^x = X^{x,0}$

Prop: The processes $\{X^{x,s}_{s+h}\}_{h\geq 0}$ & $\{X^{x,0}_h\}_{h\geq 0}$ have the same law (analogue of stationary increments) (Uses σ, b ind of time)

Lemma: Say $\exists C \ni |b_s(x)-b_s(y)| \leq C|x-y|$, $|\sigma_s(x)-\sigma_s(y)| \leq C|x-y|$, $|b_s(x)| \leq C(1+|x|)$, $|\sigma_s(x)| \leq C(1+|x|)$

a HW

Then weak uniqueness holds for the SDE $dX_t = b_t(X_t)\,dt + \sigma_t(X_t)\,dW_t$ ⋯ ⊛.

Pf of Prop: Let $Y_t = X^{x,s}_{s+t}$, & $Z_t = X^{x,0}_t$. Then $Z_t = x + \int_0^t b(Z_s)\,ds + \int_0^t \sigma(Z_s)\,dW_s$.

& $Y_t = X^{(x,s)}_{s+t} = x + \int_s^{s+t} b(X^{(x,s)}_r)\,dr + \int_s^{s+t} \sigma(\underset{\text{want } s+h}{X^{(x,s)}_r})\,dW_r = x + \int_0^t b(Y_h)\,dh + \int_0^t \sigma(Y_h)\,d\tilde{W}_h$

where $h = r - s$ & $\tilde{W}_h = W_{s+h} - W_s$. ∴ (Z, W) & $(Y, \tilde{W})$ are solutions to ⊛ with I.D. x.

& hence have the same law. QED.

Prop: Then $X_t^{(x,s)}$ has a modification which is continuous in x, s

Pf: ① Continuity in x: $X_t^x - X_t^y = x - y + \int_0^t (b(X_s^x) - b(X_s^y))\,ds + \int_0^t (\sigma(X_s^x) - \sigma(X_s^y))\,dW_s$

Let C be the lipschitz constant of b & σ. Then $E|X_t^x - X_t^y|^2 \leq c(d)\left(|x-y|^2 + (1+t)C^2 E\int_0^t |X_s^x - X_s^y|^2\,ds\right)$

So $\forall T > 0$, $t \leq T$, Gronwall $\Rightarrow E|X_t^x - X_t^y|^2 \leq c(d)|x-y|^2 e^{c(d)(1+T)C^2 t}$, Kolmogorov $\Rightarrow$ QED.

Prop (Semigroup) Almost surely $\forall r < s < t$, $\forall x \in \mathbb{R}^d$ $X_t(X_s(x,r), s) = X_t(x,r)$

Pf: Notation $X_s^r(x) = X_s^{(x,r)} = X_s(x,r)$. Let $Y_t = X_t^s(X_s^r(x))$.

Know $Y_t = Y_s + \int_s^t b(Y_{t'})\,dt' + \int_s^t \sigma(Y_{t'})\,dW_{t'}$ & $Y_s = X_s^r(x)$ a.s.

Also, $X_t^r(x) = \underbrace{x + \int_r^s b(X_{t'}^{x,r})\,dt' + \int_r^s \sigma(X_{t'}^{x,r})\,dW_{t'}}_{X_s^r(x)} + \int_s^t b(X_{t'}^{x,r})\,dt' + \int_s^t \sigma(X_{t'}^{x,r})\,dW_{t'}$

Strong Uniq $\Rightarrow \forall t \geq s$, $Y_t = X_t^{(x,r)}$ a.s. Continuity $\Rightarrow$ QED.

Notation Given a family $\{X^x\}_{x\in\mathbb{R}^d}$, we can consider the induced laws on $C[0,\infty)^d$ & get a family of measures $\{P^x\}$

Will interchangeably use $E^x f(X_t)$ & $E f(X_t^x)$ to denote the app expectation.

Thm: (Markov Prop) $\forall f \in C_b(\mathbb{R}^d)$, $s < t$, $E^x(f(X_t) \mid \mathcal{F}_s) = E^{X_s} f(X_{t-s})$ P^x a.s.

Thm: (Strong Markov) τ a stopping time, $P(\tau < \infty) = 1$. $E^x(f(X_{\tau+h}) \mid \mathcal{F}_{\tau+}) = E^{X_\tau} f(X_h)$ P^x a.s.

Pf: $f(X^0_{\tau+h}(x)) = f(X^\tau_{\tau+h}(X^0_\tau(x)))$. Let $g(y) = f(X^\tau_{\tau+h}(y))$. Note $X^\tau_{\tau+h}(y) = y + \int_\tau^{\tau+h} b(X_t^\tau(y))\,dt + \int_\tau^{\tau+h} \sigma(X_t^\tau(y))\,dW_t$

Let $\tilde{W}_t = W_{\tau+t} - W_\tau$. Know $\tilde{W}$ is a BM, independent of $\mathcal{F}_{\tau+}$. Hence

$X^\tau_{\tau+h}(y) = y + \int_0^h b(X^\tau_{\tau+t}(y))\,dt + \int_0^h \sigma(X^\tau_{\tau+t}(y))\,d\tilde{W}_t$. So if $Y_t = X^\tau_{\tau+t}(y)$, $dY_t = b(Y_t)\,dt + \sigma(Y_t)\,d\tilde{W}_t$.

By strong existence (& uniq), Y_t is $\mathcal{F}^{\tilde{W}}$-adapted & hence independent of $\mathcal{F}_{\tau+}$! $\Rightarrow g(y) = f(X^\tau_{\tau+h}(y)) = f(Y_h(y))$

is ind of $\mathcal{F}_{\tau+}$. ∴ $\exists \varphi_{i,j} \in \mathcal{B}(\mathbb{R}^d)$ & $\psi_{i,j} \in \mathcal{F}^{\tilde{W}}$ $\ni g(y,\omega) = \lim_{i\to\infty} \sum_{j\leq i} \varphi_{ij}(y)\,\psi_{ij}(\omega)$

& $\left|\sum_{j\leq i} \varphi_{ij}(y)\,\psi_{ij}(\omega)\right| \leq |g(y,\omega)|$. (∵ approx g by simple functions & simple fn's by rectangles).

$E^x(f(X_{\tau+h}) \mid \mathcal{F}_{\tau+}) = E(f(X^0_{\tau+h}(x)) \mid \mathcal{F}_{\tau+}) = E(g(X^0_\tau(x)) \mid \mathcal{F}_{\tau+})$

$= \lim_i E\left(\sum_j \varphi_{ij}(X^0_\tau(x))\,\psi_{ij} \mid \mathcal{F}_{\tau+}\right) = \lim_i \sum_j \varphi_{ij}(X^0_\tau(x))\,E\psi_{ij} = \lim_i E\left[\sum_j \varphi_{ij}(y)\,\psi_{ij}\right]_{y = X^x_\tau}$

$= \left[E g(y)\right]_{y = X^x_\tau} = \left[E f(Y_h(y))\right]_{y = X^x_\tau} \overset{\text{Weak Uniq}}{=} \left[E f(X_h(y))\right]_{y = X^x_\tau} = E^{X_\tau} f(X_h)$ P^x a.s. QED.

Def: Let X be a time homog. diff (i.e. $dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t$, b, σ lip & linear growth & $X_0 = x$)

Define the <u>generator</u> $Af(x) = \lim_{t\to 0^+} \frac{E^x f(X_t) - f(x)}{t}$ $\forall x \in \mathbb{R}^d$ & $\mathcal{D}(A) = \{f \mid Af(x) \text{ exists } \forall x\}$.

domain of A

Prop: If $f \in C_b^2(\mathbb{R}^d)$, $\nabla f \in L^\infty$ then $Af = \sum_i b_i \partial_i f + \frac{1}{2}\sum_{i,j} a_{ij}\,\partial^2_{ij} f$ where $a_{ij} = \sum_k \sigma_{ik}\sigma_{jk} = (\sigma\sigma^*)_{ij}$

Pf: Let $Lf = \sum b_i \partial_i f + \frac{1}{2} a_{ij}\,\partial^2_{ij} f$.

$\forall f \in C_b^2$, $df(X_t) = \sum \partial_i f(X_t)\,dX_t^{(i)} + \frac{1}{2}\sum \partial_{ij} f(X_t)\,d\langle X^{(i)}, X^{(j)}\rangle$

Note: $d\langle X^{(i)}, X^{(j)}\rangle_t = \sum \sigma^{(i,k)}\sigma^{(j,k')}\,d\langle W^{(k)}, W^{(k')}\rangle = a^{ij}(X_t)\,dt$

Hence $df(X_t) = \sum \partial_i f(X_t)\,\sigma^{(i,j)}\,dW^{(j)} + Lf(X_t)\,dt$.

∴ $\lim_{t\to 0^+} \frac{E^x f(X_t) - x}{t} = \lim_{t\to 0^+} \frac{1}{t}\underbrace{E^x \int_0^t Lf(X_s)\,ds}_{\to Lf(x)} + \underbrace{E^x \int_0^t \partial_i f(X_s)\,\sigma^{(i,j)}(X_s)\,dW_s^{(j)}}_{②}$

② : $E\int_0^t |\sigma^{(i,j)}(X_s)|^2\,|\partial_i f(X_s)|^2\,ds \leq \|\nabla f\|_\infty\,C E\int_0^t (1+|X_s|)^2\,ds < \infty$ (last time) $\Rightarrow$ ② $= 0$

QED.

Last Time: $dX = b(X)\,dt + \sigma(X)\,dW_t$; $X_s^{(x,s)} = x$. (b, σ Lip).

Generator: $Af(x) = \lim_{t\to 0^+} E^x \frac{f(X_t)-f(x)}{t}$. $\mathcal{D}(A) = \{ f \mid Af(x) \text{ exists } \forall x\in\mathbb{R}^d \}$.

Prop: $\forall f\in C_b^2 + Df \in L^\infty \Rightarrow f\in\mathcal{D}(A)$ & $Af = Lf$, where $L = \sum b_i \partial_i + \sum \frac{1}{2} a_{ij}\partial_{ij}$; $a_{ij} := \sum \sigma_{ik}\sigma_{jk} = (\sigma\sigma^*)_{ij}$

Pf: Itô $\Rightarrow f(X_t) - f(X_0) = \int_0^t \sum \partial_i f(X_s)\, dX_s^{(i)} + \frac{1}{2}\sum \int_0^t \partial_{ij} f(X_s)\, d\langle X^{(i)}, X^{(j)}\rangle_s$

① $dX_t^{(i)} = b^{(i)}(X_t)\,dt + \sum_k \sigma_{ik}(X_t)\, dW_t^{(k)}$, $dX_t^{(j)} = b^{(j)}(X_t)\,dt + \sum_{k'} \sigma_{jk'}(X_t)\, dW_t^{(k')}$.

$\therefore d\langle X^{(i)}, X^{(j)}\rangle = \sum_{k,k'} \sigma_{ik}(X_t)\sigma_{jk'}(X_t)\, d\langle W^{(k)}, W^{(k')}\rangle_t = \sum_k \sigma_{ik}(X_t)\sigma_{jk}(X_t)\,dt$.

$\therefore f(X_t) - f(X_0) = \underbrace{\int_0^t \sum_i \partial_i f(X_s)\, b^{(i)}(X_s)\,ds + \frac{1}{2}\int_0^t \sum_{i,j} \partial_{ij} f(X_s)\, a_{ij}(X_s)\,ds}_{\to Lf(X_s)} + \sum_k \int_0^t \partial_i f(X_s)\,\sigma_{ik}(X_s)\, dW_s^{(k)}$

Recall, Gronwall $\Rightarrow EX_t^2 \le EX_0^2 \exp(c(d)(1+T)C^2 t)$ $\forall t\le T$, where $c(d)$ is dimensional & $|b(x)|, |\sigma(x)| \le C(1+|x|)$. $\Rightarrow E\int_0^T |\partial_i f(X_s)\,\sigma_{ik}(X_s)|^2\,ds \le \|\partial_i f\|_\infty E\int_0^T C^2(1+|X_s|^2)\,ds < \infty$.

$\Rightarrow \lim_{t\to 0^+} \frac{1}{t}\left(E^x f(X_t) - f(X_0)\right) = b_i(x)\,\partial_i f(x) + \frac{1}{2} a_{ij}(x)\,\partial_{ij}^2 f(x) + 0$ QED

Thm (Dynkin's Formula): Let $f\in C_c^2(\mathbb{R}^d)$, τ a stopping time with $E\tau<\infty$. Then $E^x f(X_\tau) = f(x) + E\int_0^\tau Af(X_s)\,ds$.

Pf: By Itô, $\forall t<\infty$, $f(X_t) = f(x) + \int_0^t Lf(X_s)\,ds + \sum_{i,j}\int_0^t \partial_i f(X_s)\,\sigma_{ij}(X_s)\, dW_s^{(j)}$, a.s. Put $t = \tau(\omega)$.

$\therefore$ a.s. $f(X_\tau) = f(x) + \int_0^\tau Af(X_s)\,ds + \sum_j \int_0^\tau g_s^{(j)}\, dW_s^{(j)}$, where $g_s^{(j)} = \sum_i \partial_i f(X_s)\,\sigma_{ij}(X_s)$

NTS $E^x \int_0^\tau g_s^{(j)}\, dW_s^{(j)} = 0$. ① Know $\forall t$, $E\int_0^t |g_s^{(j)}|^2\,ds < \infty$. Hence, OST $\Rightarrow E\int_0^{\tau\wedge n} g_s^{(j)}\, dW^{(j)} = 0$.

② $g^{(j)}$ is bounded $\forall j$ ($\because f\in C_c$ & σ is locally bdd). Say $|g^{(j)}|\le C$.

③ $E^x\left(\int_{\tau\wedge n}^\tau g_s^{(j)}\, dW_s^{(j)}\right)^2 = E^x\left(\int_0^\infty \chi_{\{\tau\wedge n\le s\le\tau\}}\, g_s^{(j)}\, dW^{(j)}\right)^2 = E^x\int_0^\infty \chi_{\{\tau\wedge n\le s\le \tau\}} |g_s^{(j)}|^2\,ds$

$= E\int_{\tau\wedge n}^\tau |g^{(j)}|^2\,ds \le C^2 E(\tau - \tau\wedge n) \to 0$. QED.

Remark: The assumption $f\in C_c^2(\mathbb{R}^d)$ can be relaxed, as long as one can still conclude $E\int_{\tau\wedge n}^\tau g^{(j)}\, dW^{(j)} \to 0$.

Eg: $X_t = W_t$. τ = exit time from B_R (ball of radius R). Compute $E\tau$.

Pf: Let $f(x) = |x|^2$ in B_R & some C_c^2 extension outside. Then $Af = \frac{1}{2}\Delta f$, & $\forall n\in\mathbb{N}$, Dynkin $\Rightarrow$

$E^x f(W_{\tau\wedge n}) = f(x) + E^x\int_0^{\tau\wedge n} Af(X_s)\,ds \Rightarrow R^2 \ge |x|^2 + E^x\int_0^{\tau\wedge n} d\,ds = |x|^2 + dE^x(\tau\wedge n)$

$\Rightarrow E^x\tau \le \frac{1}{d}(R^2 - |x|^2) < \infty$. By Dynkin, $R^2 = E^x f(W_\tau) = |x|^2 + d\,E^x\tau \Rightarrow E^x\tau = \frac{R^2 - x^2}{d}$. QED

Recurrence of BM: Let $d\ge 2$, $R>0$, $x\in\mathbb{R}^d$ with $|x|>R$. Let τ = hitting time of BM to B_R

(i.e. $\tau = \tau(\omega) = \inf\{t\ge 0 \mid W_t^x \in B_R\}$) Compute $P(\tau^x < \infty)$.

① Find radial solutions of $\Delta u = 0$: If $u(x) = f(|x|)$, $\Delta u = f''(|x|) + f'(|x|)\frac{(d-1)}{|x|}$.

Solve the ODE $f''(r) + f'(r)\frac{(d-1)}{r} = 0$. Let $f(x) = \begin{cases} \ln|x| & d=2 \\ |x|^{2-d} & d>2 \end{cases}$.

② Let $A_n = \{x \mid R < |x| < nR\}$, & τ_n = exit time from A_n. Note $\tau_n \le \tau_{B_{nR}} \Rightarrow E\tau_n < \infty$.

Choose $u(x) = \begin{cases} \ln|x| & d=2 \\ |x|^{2-d} & d>2 \end{cases}$ in A_n & some C_c^2 outside A_n.

Dynkin $\Rightarrow E^x u(W_{\tau_n}) = u(x) + E^x\int_0^{\tau_n} Au(W_s)\,ds = u(x) + 0$.

Case I: $d=2$: Let $p_n = P(|W_{\tau_n}| = R)$, $q_n = 1 - p_n = P(|W_{\tau_n}| = nR)$.

Then $\ln|x| = p_n \ln R + q_n \ln(nR) \Rightarrow \lim_{n\to\infty} q_n = 0 \Rightarrow \lim_{n\to\infty} p_n = 1$

Recurrent in the sense of Harris.

Note $\{\tau<\infty\} = \overset{\text{nested}}{\bigcup_n} \{|W_{\tau_n}| = R\}$ $\therefore P^x\{\tau<\infty\} = \lim_{n\to\infty} P\{|W_{\tau_n}| = R\} = \lim_{n\to\infty} p_n = 1$

Case II: $d>2$: p_n, q_n as above. $|x|^{2-d} = p_n|R|^{2-d} + q_n|nR|^{2-d}$, conv to 0 $\Rightarrow \lim_{n\to\infty} p_n = \left(\frac{R}{|x|}\right)^{d-2}$

$\Rightarrow P(\tau<\infty) = \left(\frac{R}{|x|}\right)^{d-2} < 1$ (B.M. is transient).

Kolmogorov backward eq: Let $f\in C_c^2$, & define $u(x,t) = E^x f(X_t)$. Then $\forall t$, $u(\cdot,t)\in D(A)$, is cts in time,

& u solves $\partial_t u - Au = 0$ with $u(x,0) = f(x)$.

Pf: ① $u(x,t+h) = E^x u(X_h, t)$: Pf: $E^x f(X_{t+h}) = E^x E^x(f(X_{t+h}) \mid \mathcal{F}_h) = E^x E^{X_h} f(X_t) = E^x u(X_h, t)$

② $Au_t = \lim_{h\to 0^+} \frac{E^x u(X_h,t) - u(x,t)}{h} = \lim_{h\to 0^+} \frac{u(x,t+h) - u(x,t)}{h} = \partial_t u$, provided it exists.

③ $u(x,t) = E^x f(X_t) = f(x) + E^x\int_0^t Af(X_s)\,ds \Rightarrow \lim_{h\to 0^+} \frac{u(x,t+h)-u(x,t)}{h}$ exists ($= E^x Af(X_t)$). QED.

Remark: Can weaken $f\in C_c^2$ to $f\in C_b^2$, (or even poly growth). [Only used in step ③ above].

Conversely: If $u\in C_b^{2,1}(\mathbb{R}^d\times[0,\infty))$ & $\partial_t u = Lu$ with $u(x,0) = f(x)$, Then $u(x,t) = E^x f(X_t)$.

Cor: (Maximum Principle) Say $u\in C_b^{2,1}(\mathbb{R}^d\times[0,\infty))$ & $\partial_t u - Lu = 0$. Then $\sup_{x,t} u(x,t) = \sup_x f(x)$.

Pf: Know $u(x,t) = E^x f(X_t) \Rightarrow |u(x,t)| \le \sup_x f(x)$ QED

Recall: $dX = b(X)\,dt + \sigma(X)\,dW_t$, $a_{ij} = \sum_k \sigma_{ik}\sigma_{jk}$. $Af(x) = \lim_{t\to 0} \frac{1}{t} E^x(f(X_t) - f(x))$

$\mathcal{L}f(x) = \sum_i b_i(x)\,\partial_i f(x) + \frac{1}{2}\sum_{i,j} a_{ij}(x)\,\partial^2_{ij} f(x)$; $[\, f \in C^2_b \Rightarrow Af = \mathcal{L}f \,]$.

Last time: ① Dynkin's form: $f \in C^2_c \Rightarrow Af = \mathcal{L}f$. & $f \in C^2_c$, $E^x\tau < \infty \Rightarrow E^x f(X_\tau) = f(x) + E^x\int_0^\tau Af(X_s)\,ds$.

Will instead prove F-K which is more genl.

② Kolmogorov backward eq: $f \in C^2_{\cancel{c}\,b}$, $u(x,t) = E^x f(X_t)$. Then $u(\cdot,t) \in \mathcal{D}(A)$, $\partial_t u - Au = 0$, $u(x,0) = f(x)$.

③ Converse: If $u \in C^{2,1}_b(\mathbb{R}^d \times [0,\infty))$ & $\partial_t u = \mathcal{L}u$ with $u(x,0) = f(x)$, Then $u(x,t) = E^x f(X_t)$.

Remark: In general, given the PDE $\partial_t u - \mathcal{L}u = 0$, where $\mathcal{L} = \sum b_i\partial_i + \frac{1}{2}\sum a_{ij}\partial^2_{ij}$, one needs to find a diffusion X ∋ $\mathcal{L}f = A_X f$ $\forall f \in C^2_b$ before one can use Kolmogorov/etc. For this need a matrix $\sigma = \sigma(x)$ ∋ $a(x) = (a_{ij}(x)) = \sigma(x)\sigma^*(x)$ & need σ Lip.

Thm: Say $a(x)$ is a non-neg def $d\times d$ matrix & $a \in C^2_b$, then $\exists\, \sigma$ Lip ∋ $a = \sigma\sigma^*$.

Remark: $a \in C^2 \Rightarrow \sigma$ is only locally Lip.

Remark: The assumption $a \in C^2$ is necessary: $a(x) = |x|^{2-\varepsilon}$. Then $\sigma(x) = |x|^{1-\varepsilon/2}$, which is not Lip.

Feynman-Kac Formula: Let $f \in C^2_b(\mathbb{R}^d)$, $c \in C(\mathbb{R}^d)$ is bdd below.

Implicitly $\Rightarrow u(\cdot,t) \in \mathcal{D}(A)$ $\forall t$.

① Let $u(x,t) = E^x f(X_t)\exp\left(-\int_0^t c(X_s)\,ds\right)$. Then $\partial_t u - Au + cu = 0$ & $u(x,0) = f(x)$.

② Conversely, Say $u \in C^{2,1}(\mathbb{R}^d \times [0,\infty))$ & $\partial_t u - \mathcal{L}u + cu = 0$, with $u(x,0) = f(x)$. Then

Partial Converse because $C^{2,1}_c \subseteq \mathcal{D}(A)$

$$u(x,t) = E^x f(X_t)\exp\left(-\int_0^t c(X_s)\,ds\right)$$

Pf: ②: Let $Y_t = u_{T-t}(X_t)\exp\left(-\int_0^t c(X_s)\,ds\right)$.

$$dY = \exp(\,)\Big[\cancel{-\partial_t u_{T-t}(X_t)\,dt} + \cancel{\mathcal{L}u_{T-t}(X_t)\,dt} + \sum \partial_i u_{T-t}(X_t)\,\sigma_{ij}(X_t)\,dW^{(j)}_t\Big)$$

$$+\, u_{T-t}(X_t)\exp(\,)\,(\cancel{-c(X_t)})\,dt = \exp(\,)\sum \partial_i u \cdots dW.$$

$\Rightarrow Y \in \mathcal{M}_{c,loc}$. Also f bdd, c bdd below $\Rightarrow u$ bdd $\Rightarrow Y$ bdd $\Rightarrow Y \in \mathcal{M}_c$.

$\therefore E^x Y_T = E^x Y_0$. Compute $E^x Y_T = E^x f(X_T)\exp\left(-\int_0^T c(X_s)\,ds\right)$ & $E^x Y_0 = E^x u_T(X_0)\exp(0) = u_T(x)$

QED.

① Claim: $u(x,t+h) = E^x u_t(X_h)\exp\left(-\int_0^h c(X_s)\,ds\right)$.

Pf: $u(x,t+h) = E^x f(X_{t+h})\exp\left(-\int_0^{t+h} c(X_s)\,ds\right) = E^x E^x(\quad | \mathcal{F}_h)$

$$= E^x\left[\exp\left(-\int_0^h c(X_s)\,ds\right) E^x\left(f(X_{t+h})\exp\left(-\int_h^{t+h} c(X_s)\,ds\right)\Big|\mathcal{F}_h\right)\right]$$

$$= E^x\left[\exp\left(-\int_0^h c(X_s)\,ds\right) E^{X_h} f(X_t)\exp\left(-\int_0^t c(X_s)\,ds\right)\right] = E^x u_t(X_h)\exp\left(-\int_0^h c(X_s)\,ds\right)$$ QED.

Now $\partial_t u_t(x) = \lim_{h\to 0^+} \frac{u(x,t+h) - u(x,t)}{h} = \lim_{h\to 0^+} \frac{1}{h} E^x\left[u_t(X_h)\exp\left(-\int_0^h c(X_s)\,ds\right) - u_t(x)\right]$

$$= \lim_{h\to 0^+}\frac{1}{h}E^x(u_t(X_h) - u_t(x)) + \lim_{h\to 0^+}\frac{1}{h}E^x u_t(X_h)\left(\exp\left(-\int_0^h c(X_s)\,ds\right) - 1\right)$$

$$= Au_t(x) - cu_t(x)$$ QED.

Say now the diffusion has a smooth transition density $p(x,s,y,t)\,dy = P(X^{(x,s)}_t \in dy)$

X time homog $\Rightarrow p(x,s;y,t) = p(x,0;y,t-s)$. Also, given $f \in C^2_b$,

$u(x,t) = E^x f(X_t) = \int f(y)\,p(x,0,y,t)\,dy$. Since $\partial_t u = \mathcal{L}u$ $\forall f$ (Kolmogorov Backward)

$\Rightarrow \partial_t p = \mathcal{L}_x p \Leftrightarrow \partial_s p + \mathcal{L}_x p = 0$. [PDE for p in the <u>initial</u> variables x & s].

Also, $\lim_{s\to t^-} p(x,s;\cdot,t) = \delta_x$ follows quickly from non-degeneracy of the diffusion X.

So Kolmogorov Backward can be restated as $u(x,s) = E^{(x,s)} f(X_T)$, then

$\partial_s u \;\underline{\underline{+}}\; \mathcal{L}u = 0$, with final condition $u(x,T) = f(x)$.

Kolmogorov Forward Eq: Equation for p in the final variables y & t.

Let $\mathcal{L}^*_y(g) = \sum -\frac{\partial}{\partial y_i}(b^{(i)}_t g) + \frac{1}{2}\sum \frac{\partial^2}{\partial y_i \partial y_j}(a_{ij}\,g)$.

If p exists & is smooth, then $\partial_t p - \mathcal{L}^*_y p = 0$, with $\lim_{t\to s^+} p(\cdot,s,y,t) = \delta_y$

Remk: $\mathcal{L}^* = L^2$ dual of $\mathcal{L}$: $\forall f,g \in C^2_0$ $\int_{\mathbb{R}^d}(\mathcal{L}f)\,g = \int_{\mathbb{R}^d} f\,(\mathcal{L}^* g)$

Pf: Fix $T > 0$, $f \in C^2_0$, & $u(x,s) = E^{(x,s)} f(X_T)$. Know $\partial_s u + \mathcal{L}u = 0$ for $s < T$ & $u(x,T) = f(x)$.

Markov $\Rightarrow u(x,s) = E^{(x,s)} u_t(X_t) = \int_{\mathbb{R}^d} p(x,s;y,t)\,u(y,t)\,dy$ $(s \le t \le T)$.

(need $p \to 0$ at ∞)

diff wrt t: $0 = \int_{\mathbb{R}^d} \partial_t p\, u + p\,\partial_t u = \int \partial_t p\, u - p\,\mathcal{L}u = \int(\partial_t p - \mathcal{L}^* p)\,u$ $\forall t \le T$

Put $t = T$: $\int_{\mathbb{R}^d}(\partial_t p(x,s;y,T) - \mathcal{L}^*_y p(x,s;y,T))\,f(y)\,dy = 0$ $\forall f \in C^2_b \Rightarrow \partial_t p - \mathcal{L}^*_y p = 0$. QED.

Dirichlet - Poisson Problems. Let b, σ be time indep, Lip. $L = \sum b_i \partial_i + \frac{1}{2}\sum a_{ij}\partial^2_{ij}$. Let X^x solve $dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t$ with $X_0^x = x$ a.s. Let $D \subseteq \mathbb{R}^d$ be a domain, $\tau = \inf\{t \mid X_t \notin D\}$.

Prop: Say $u \in C_b^2(D)$, $g : D \to \mathbb{R}$ is $\ni E\left|\int_0^\tau g(X_s)\,ds\right| < \infty$. Then if $-Lu = g$ in D & $\lim_{t\to\tau} u(X_t) = f(X_\tau)\chi_{\{\tau<\infty\}}$ we must have $u(x) = E^x\left(f(X_\tau)\chi_{\{\tau<\infty\}} + \int_0^\tau g(X_s)\,ds\right)$

Remk: If $\tau < \infty$ a.s. & $u \in C_b^2(D) \cap C(\bar{D})$ then $\lim_{t\to\tau^-} u(X_t) = f(X_\tau)\chi_{\{\tau<\infty\}} \iff u|_{\partial D} = f$.

If $P(\tau = \infty) > 0$, then the B.C. on u is a "vanish at ∞" condition.

Pf: Let $D = \cup D_n$, where $\bar{D}_n \subset D_{n+1}$, D_n open, bdd. $\tau_n = \inf\{t \mid X_t \notin D_n\} \wedge n$. Then Dynkin $\Rightarrow$

$E^x u(X_{\tau_n}) = u(x) + E^x \int_0^{\tau_n} Au(X_s)\,ds \Rightarrow u(x) = E^x u(X_{\tau_n}) + E^x\int_0^{\tau_n} g(X_s)\,ds$ & send $n \to \infty$ QED.

Caveats: ① How solutions to $-Lu = g$ attain the boundary value is delicate. (i.e. assuming $u \in C_b^2(D) \cap C(\bar{D})$ is too much)

② Unfortunately $u(x) = E^x f(X_\tau)\chi_{\{\tau<\infty\}}$ need not even be continuous up to ∂D

Lemma Let $D = B_R \subseteq \mathbb{R}^d$. Say $\exists \lambda > 0 \ni |\sigma^T(x) y| \geq \lambda |y|$ $\forall x \in D$, $y \in \mathbb{R}^d \Rightarrow E^x \tau_D < \infty$ ($\tau_D = \inf\{t \mid X_t \notin D\}$).

Pf: PDE $\Rightarrow \exists u \in C_b^2(D) \cap C(\bar{D}) \ni -Lu = 1$ in D & $u = 0$ on ∂D. Dynkin $\Rightarrow$

$E^x u(X_{\tau\wedge n}) = u(x) + E^x \int_0^{\tau \wedge n} Lu(X_s)\,ds \Rightarrow E^x(\tau \wedge n) = u(x) - E^x u(X_{\tau\wedge n}) \leq 2\|u\|_\infty$ & send $n \to \infty$. QED.

(u(x) by M.P.)

Cor: σ u. elliptic & $U \subseteq \mathbb{R}^d$ bdd $\Rightarrow E^x \tau_U < \infty$. ($\tau_U = \inf\{t \mid X_t \notin U\}$)

$\iff u(X^\tau) \in \mathcal{M}_c$

& u cts.

Def: u is called X-harmonic in D if $\forall$ stopping times $\tau \leq \tau_D$, $u(x) = E^x u(X_\tau)$. (u is X-sub harmonic if $u(x) \leq E^x u(X_\tau)$ & ~~u is upper s.c.~~ $f(x) \geq \varlimsup_{y\to x} f(y)$. $\|^{ly}$ X-super harmonic if $u(x) \geq E^x u(X_\tau)$ & ~~u is lower s.c.~~ $u(x) \leq \varliminf_{y\to x} u(y)$)

Prop: If $u \in C_b^2(D)$ & $Au \geq 0$ then u is X-sub h. in D. ($\|^{ly}$, $Au \leq 0 \Rightarrow$ super h.)

Pf: Let $\tau \leq \tau_D$. $E^x u(X_{\tau\wedge n}) = u(x) + E^x \int_0^{\tau\wedge n} Lu(X_s)\,ds \geq u(x)$. & send $n \to \infty$. QED.

Prop: Let $f : \partial D \to \mathbb{R}$ be Borel, bdd. Let $u(x) = E^x f(X_{\tau_D})$. Then (u cts $\Rightarrow$) u is X-harmonic. (called the X-harm ext of u)

& $\lim_{t\to\tau_D^-} u(X_t) = f(X_{\tau_D})$ P^x a.s.

Pf: Let $\sigma < \tau_D$. $u(x) = E^x E^x(f(X_{\tau_D}) \mid \mathcal{F}_{\sigma^+}) \underset{\text{Claim.}}{=} E^x E^{X_\sigma} f(X_{\tau_D}) = E^x u(X_\sigma) \Rightarrow u$ is X-harmonic.

Pf of claim: $\Omega = [C([0,\infty))]^d$. $X_t(\omega) = \omega(t)$. $\theta_t(\omega)(s) = \omega(s+t)$

Then $f(X_{\tau_D}) = f \circ X_{\tau_D} \circ \theta_\sigma \Rightarrow E^x(f(X_{\tau_D}) \mid \mathcal{F}_{\sigma^+}) = E^x(f \circ X_{\tau_D} \circ \theta_\sigma \mid \mathcal{F}_{\sigma^+}) = E^{X_\sigma} f(X_{\tau_D})$. QED.

For B.C. Let $D = \cup D_n$, D_n open, $\bar{D}_n \subseteq D_{n+1}$. Let $\tau_n = \tau_{D_n}$, $\mathcal{G}_n = \mathcal{F}_{\tau_n}$, $M_n = u(X_{\tau_n})$.

Then $E^x(f(X_{\tau_D}) \mid \mathcal{F}_{\tau_n}) = E^{X_{\tau_n}} f(X_{\tau_D}) = u(X_{\tau_n}) = M_n \Rightarrow \{M_n, \mathcal{G}_n\}$ a martingale.

$M_n \leq \|f\|_{L^\infty(\partial D)}$ & Doob $\Rightarrow M_n \to M_\infty = f(X_{\tau_D})$ a.s. & in L^p $\forall p < \infty$.

$\forall n$, Let $N_t = u(X_{\tau_n \vee (\tau_{n+1}\wedge t)}) - u(X_{\tau_n})$. Note $\{N_t, \mathcal{F}_{\tau_n \wedge t}\} \in \mathcal{M}_c$

Claim: $\forall \sigma < \tau_D$, $Y_t = u(X_{\sigma\wedge t})$. Then $\{Y_t, \mathcal{F}_{\sigma\wedge t}\}$ is a cts mg.

Pf: $u(X_{\sigma\wedge t}) = E^{X_{\sigma\wedge t}} f(X_{\tau_D}) = E(f(X_{\tau_D}) \mid \mathcal{F}_{\sigma\wedge t^+})$, from above. Tower + cts $\Rightarrow$ QED.

Claim $\Rightarrow N \in \mathcal{M}_c$

$\therefore P\left(\sup_{\tau_n \leq t \leq \tau_{n+1}} |u(X_t) - u(X_{\tau_n})| > \varepsilon\right) \leq \frac{1}{\varepsilon^2} E|u(X_{\tau_{n+1}}) - u(X_{\tau_n})|^2 = \frac{1}{\varepsilon^2} E|M_{n+1} - M_n|^2 \to 0$ QED.

Def: Let $\tau_D = \inf\{t > 0 \mid X_t \notin D\}$. We say $x \in \partial D$ is regular if $\tau_D = 0$ P^x a.s. (Note $P^x(\tau_D = 0) \in \{0, 1\}$).

Thm: Let σ u. elliptic, $f : \partial D \to \mathbb{R}$ cts, bdd, $u(x) = E^x f(X_{\tau_D})$. Then $u \in C^{2+\alpha}(D)$, $-Lu = 0$ in D

& $\forall x \in \partial D$ regular $\lim_{y\to x} u(y) = f(x)$.

Prop: Let σ u. elliptic & $x_0 \in \partial D$. If $\exists$ a cone based at x_0 which is locally outside D, then x_0 is regular.