SDE's.

Let $d \in \mathbb{N}$, $b : \mathbb{R}^d \times [0,\infty) \longrightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times [0,\infty) \longrightarrow \mathbb{R}^{d^2}$, Let $(\Omega, \mathcal{F}, P)$ be a prob space, W a B.M., ξ a R.V. Let $\mathcal{F}_t = \sigma\left(\bigcup_{s\le t}\sigma(W_s) \cup \sigma(\xi) \cup N\right)$ $(N = \{\text{null sets}\})$.

Def: We say X is a <u>strong solution</u> to the SDE $dX_t = b_t(X_t)\,dt + \sigma_t(X_t)\,dW_t$, (*) with initial data ξ if (1) $X_0 = \xi$ a.s., (2) X_t is $\mathcal{F}_t$ meas, (3) $\int_0^t |b_s(X_s)|^2 ds < \infty$ a.s. & $\int_0^t \sigma_s^{(i,j)}(X_s)^2\, ds < \infty$ a.s. & (4) $X_t^{(i)} = \xi^{(i)} + \int_0^t b_s^{(i)}(X_s)\,ds + \sum_j \int_0^t \sigma_s^{(i,j)}(X_s)\,dW_s^{(j)}$ a.s.

Remark: SDE's arose through the study of diffusions: Processes with given spatial mean $\lim_{t\to 0}\frac{1}{t}(E^x X_t - x)$ & covariances $\lim_{t\to 0}\frac{1}{t}E^x(X_t^{(i)} - x^{(i)})(X^{(j)} - x^{j})$. Can easily check that for solutions to (*), $b(x) = \lim_{t\to 0}\frac{1}{t}E^x(X_t - x)$ & $\sum_k \sigma_{ik}(x)\sigma_{jk}(x) = \lim_{t\to 0}\frac{1}{t}E^x(X_t^{(i)} - x^{(i)})(X_t^{(j)} - x^{j})$

Def: We say strong uniqueness holds for the SDE (*) if any two strong solutions (with the same initial data) are indistinguishable.

Thm. (Uniqueness) Say $\exists\, C \ni \forall t$, $|b_t(x) - b_t(y)| \le C|x-y|$ & $|\sigma_t^{(ij)}(x) - \sigma_t^{(ij)}(y)| \le C|x-y|$. $\forall x,y$ & $|\sigma^{(i,j)}(x)| \le C$ $\forall x$, then strong uniqueness holds for (*).

Lemma (Gronwall) If $y(t)$ is some fn $\ni y_t \le a + \int_0^t b_s y_s\,ds$, where $b \ge 0 \Rightarrow y_t \le a e^{\int_0^t b_s ds}$.

Pf: Let $z_t = a + \int_0^t b_s y_s\,ds$. Then $\dot z_t = b_t y_t \le b_t z_t \Rightarrow \frac{d}{dt}\left(e^{-\int_0^t b} z_t\right) = e^{-\int_0^t b}(\dot z_t - b_t z_t) \le 0$.

$\Rightarrow e^{-\int_0^t b} z_t \le z_0 \Rightarrow y_t \le z_t \le z_0 e^{\int_0^t b} = a e^{\int_0^t b}$. QED.

Pf (Uniqueness) Say X & Y are 2 strong solutions to (*), with $X_0 = Y_0$ a.s. Put $Z = X - Y$. Then

$Z_t^{(i)} = \int_0^t \left(b_s^{(i)}(X_s) - b_s^{(i)}(Y_s)\right) ds + \sum_j \int_0^t \left(\sigma_s^{(ij)}(X_s) - \sigma_s^{(ij)}(Y_s)\right) dW_s^{(j)}$.

(Let $c_1 = c_1(d, C)$ be a constant that changes from line to line)

$\Rightarrow E|Z_t^{(i)}|^2 \le c_1 E\left(\int_0^t |b_s(X) - b_s(Y)|\,ds\right)^2 + c_1 \sum E\left(\int_0^t \sigma_s^{(i,j)}(X_s) - \sigma_s^{(i,j)}(Y_s)\, dW_s^{(j)}\right)^2$

$\le c_1 t E\int_0^t |X_s - Y_s|^2 ds + c_1 E\int_0^t |X_s - Y_s|^2 = c_1(1+t)\int_0^t E|Z_s|^2\,ds$

$\Rightarrow \forall t \le T$, $E|Z_t|^2 \le c_1(1+T)\int_0^t E|Z_s|^2\,ds \Rightarrow E|Z_t|^2 \le 0\, e^{\int_0^t (\,)} = 0$. QED.

Remark: By localizing, one can show strong uniqueness holds for (*) if we assume $\forall n\ \exists C_n \ni$

$\sup_t\left(|\sigma_t(x) - \sigma_t(y)| + |b_t(x) - b_t(y)|\right) \le C_n|x-y|$ whenever $|x| \le n$ & $|y| \le n$. (locally Lip)

Thm (Existence) Say $\forall t \ge 0$, $x,y \in \mathbb{R}^d$, $|b_t(x) - b_t(y)| \le C|x-y|$, $|\sigma_t(x) - \sigma_t(y)| \le C|x-y|$ & $|b_t(x)| \le C(1+|x|)$, $|\sigma_t(x)| \le C(1+|x|)$, then (*) has a strong solution, which is cts in time.

Remark: Linear growth is necessary to prove existence of global in time solutions, even for ODE's.

Pf: Let $X^{(0)} = \xi$, $X_t^{(n+1)} = \xi + \int_0^t b_s(X_s^{(n)})\,ds + \int_0^t \sigma_s(X_s^{(n)})\,dW_s$ Notation: $\left(\int_0^t \sigma_s\,dW_s\right)^{(i)} = \sum_j \int_0^t \sigma_s^{(i,j)}\,dW_s^{(j)}$

Claim: Let $T > 0$. $\exists\, c_2 = c_2(d, c, T) \ni \forall t \le T$, $E\left(X_t^{(n+1)} - X_t^{(n)}\right)^2 \le \frac{(c_2 t)^{n+1}}{(n+1)!}$

Pf: For $n = 1$, $X^{(1)} - X^{(0)} = \int_0^t b_s(\xi)\,ds + \int_0^t \sigma_s(\xi)\,dW_s \Rightarrow E(X_t^{(1)} - X_t^{(0)})^2 \le c_1(d)(c^2t^2 + c^2 t) \le \underbrace{c_1(d)\,\tilde c^2(1+T)}_{c_2}\,t$ $\cdot E|\xi|^2 =]$

For $n > 1$, $X_t^{(n+1)} - X_t^{(n)} = \int_0^t \left(b_s(X_s^{(n)}) - b_s(X_s^{(n-1)})\right) ds + \int_0^t \left(\sigma_s(X_s^{(n)}) - \sigma_s(X_s^{(n-1)})\right) dW_s$.

$E\int_0^T |\sigma_s(X^{(n)}) - \sigma_s(X^{(n-1)})|^2 ds \le c^2 \int_0^T E|X^{(n)} - X^{(n-1)}|^2 ds \le c^2 \frac{c_2^n T^{n+1}}{(n+1)!} < \infty$.

$\Rightarrow E\left|\int_0^t \left(\sigma_s(X_s^{(n)}) - \sigma_s(X_s^{(n-1)})\right) dW_s\right|^2 = E\int_0^t |\sigma_s(X_s^{(n)}) - \sigma_s(X_s^{(n-1)})|^2 ds$ by Itô.

$\Rightarrow E|X_t^{(n+1)} - X_t^{(n)}|^2 \le c_1(d)\left(c^2 t \int_0^t E|X_s^{(n)} - X_s^{(n-1)}|^2 ds + c^2 \int_0^t E|X_s^{(n)} - X_s^{(n-1)}|^2 ds\right)$

$\le c_1(d)\, c^2(1+T)\int_0^t \frac{(c_2 s)^n}{n!}\,ds = \frac{c_2^{n+1} t^{n+1}}{(n+1)!} \Rightarrow$ QED (Claim 1)

$\therefore \forall t$, $(X_t^{(n)})$ is Cauchy in $L^2(\Omega, \mathcal{F}_t)$. $\Rightarrow \exists\, X_t \in L^2(\Omega, \mathcal{F}_t) \ni (X_t^{(n)}) \longrightarrow X_t$ in $L^2(\Omega, \mathcal{F}_t)$.

Note that $\sup_{t\le T} E|X_s^{(n)} - X_s^{(n-1)}|^2 \le \frac{(c_2 T)^n}{n!} \Rightarrow \sup_{s \le T} E|X_s^{(n)} - X_s|^2 \longrightarrow 0$.

$\Rightarrow \int_0^T E|X_s^{(n)} - X_s|^2 ds \longrightarrow 0 \Rightarrow (X^{(n)}) \longrightarrow X$ in $\mathcal{L}_T(W) \Rightarrow \forall t \le T$,

$E\left[\int_0^t \sigma_s(X_s^{(n)})\,dW_s - \int_0^t \sigma_s(X_s)\,dW_s\right]^2 \le c' E\int_0^t |X_s^{(n)} - X_s|^2 ds \longrightarrow 0$

& $E\left[\int_0^t b_s(X_s^{(n)})\,ds - \int_0^t b_s(X_s)\,ds\right]^2 \le c' t E\int_0^t |X_s^{(n)} - X_s|^2 ds \longrightarrow 0$

$\Rightarrow X_t = \xi + \int_0^t b_s(X_s)\,ds + \int_0^t \sigma_s(X_s)\,dW_s$ a.s., $\forall t$

For time continuity, note X is (jointly) meas. So now let $Y_t = \xi + \int_0^t b_s(X_s)\,ds + \int_0^t \sigma_s(X_s)\,dW_s$,

Then $E\int_0^T |b_s(X_s) - b_s(Y_s)|^2 ds = \int_0^T E|\quad|^2 = 0 \Rightarrow \int_0^t |b_s(X_s) - b_s(Y_s)|^2 \le \int_0^T (\,) = 0$ a.s.

$\Rightarrow$ a.s., $\forall t \le T$, $\int_0^t b_s(X_s)\,ds = \int_0^t b_s(Y_s)\,ds$. & ||^{ly} $\int_0^t \sigma_s(X_s)\,dW_s = \int_0^t \sigma_s(Y_s)\,dW_s$.

$\Rightarrow Y_t = \xi + \int_0^t b_s(Y_s)\,ds + \int_0^t \sigma_s(Y_s)\,dW_s$. Finally since T is arbitrary $\Rightarrow$ QED.

Def: We say $X, W, \Omega, \mathcal{F}, P; \{\mathcal{F}_t\}$ is a weak solution to $dX_t = b_t(X_t)\,dt + \sigma_t(X_t)\,dW_t$ if

① X_t is $\mathcal{F}_t$ meas, ② $\int_0^t |b_s(X_s)|^2 ds < \infty$ a.s. & $\int_0^t |\sigma_s(X_s)|^2 ds < \infty$ a.s.

& ③ $X_t = X_0 + \int_0^t b_s(X_s)\,ds + \int_0^t \sigma_s(X_s)\,dW_s$ a.s.

Remark: Strong solutions, $\Omega, \mathcal{F}, P$ & W given, $\{\mathcal{F}_t\}$ the (augmented) filtration of W & the I.D.

Weak solutions, $\Omega, \mathcal{F}, P$ & more imp, W, $\{\mathcal{F}_t\}$ are part of the solution & can be chosen as needed.

Def (Uniqueness) ① Strong uniqueness: X, Y two strong solutions, $X_0 = Y_0$ a.s. $\Rightarrow$ X & Y are indistinguishable

* ② Pathwise uniqueness: X, Y two (possibly weak) solutions on the same prob space (w.r.t. the same B.M & filtration) then $X_0 = Y_0$ a.s. $\Rightarrow$ X & Y are indistinguishable.

③ Weak uniqueness: If X & $\tilde{X}$ are two (weak) solutions & $\forall A \in \mathcal{B}(\mathbb{R}^d)$, $P(X_0 \in A) = \tilde{P}(Y_0 \in A)$

Then X & $\tilde{X}$ have the same laws (i.e. $P((X_{t_0}, \dots X_{t_n}) \in A) = \tilde{P}((\tilde{X}_{t_0}, \dots \tilde{X}_{t_n}) \in A)$ $\forall n, A, t_0 \dots t_n$.

Remark: Strong existence $\Rightarrow$ Weak existence & the converse is false.

Thm (Yamada Watanabe) ① Weak existence & pathwise uniqueness $\Rightarrow$ Strong existence.

② Strong Uniqueness $\Rightarrow$ Weak uniqueness. (Converse is of course false).

Eg (Tanaka) The SDE $dX = \operatorname{sign}(X)\,dW$, with initial data 0 has weak existence & uniqueness, but not strong exist or uniq. (Here $\operatorname{sign}(x) = \operatorname{sign}_-(x) = \begin{cases} 1 & x > 0 \\ -1 & x \le 0 \end{cases}$ for convenience)

① If X is a (weak) solution, then $d\langle X\rangle = \operatorname{sign}(X)^2 dt = dt$. Levy $\Rightarrow X$ is a BM $\Rightarrow$ weak uniqueness.

② If X is a weak solution, let $Y = -X$. $Y_t = -X_t = \int_0^t \operatorname{sign}_+(-X_t)\,dW_t$. Since X is a B.M.,

$|\{t \mid X_t = 0\}| = 0$ a.s. $\Rightarrow \int_0^t [\operatorname{sign}_+(X_t) - \operatorname{sign}_-(X_t)]^2 dt = 0$ a.s. $\Rightarrow dY_t = -\operatorname{sign}_-(Y_t) \Rightarrow$ no strong uniq.

③ Let X be a B.M. Let $W_t = \int_0^t \operatorname{sign}(X_s)\,dX_s$. Levy $\Rightarrow d\langle W\rangle_t = \operatorname{sign}(X_s)^2 dt = dt$

$\Rightarrow W$ is a BM. Also, $dW_s = \operatorname{sign}(X_s)\,dX_s \Rightarrow dX_s = \operatorname{sign}(X_s)\,dW_s \Rightarrow$ Weak existence.

④ Say X is a strong solution. Since X is a BM, by Tanaka,

$|X_t| = \int_0^t \operatorname{sign}(X_s)\,dX_s + L_t(X)$, where $L_t(X) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} |\{s \in [0,t] \mid |X_s| < \varepsilon\}|$

$\Rightarrow |X_t| = W_t + L_t(X) \Rightarrow W_t = |X_t| - L_t(X)$. $\Rightarrow W_t$ is adapted to $\{\mathcal{F}_t^{|X|}\} \subsetneq \{\mathcal{F}_t^X\} \subseteq \{\mathcal{F}_t^W\} \Rightarrow\Leftarrow$.

Thm: Say $\exists C \ni |b_s(x)| \le C$ $\forall x \in \mathbb{R}^d, s \le T$. Then $dX_t = b_t(X_t)\,dt + dW$ has weak existence & uniqueness $\forall$ initial distributions μ.

Idea: X a B.M. $W = -\int_0^t b_s(X_s)\,ds + X_t$ ← Change measure & make W a B.M. Now measure $dX = b(X_s)ds + dW$!

Pf: ① Existence: Let $(X, \mathcal{F}_t, \{P^x\}_{x \in \mathbb{R}^d})$ be a Brownian family. Let $Z_t = \exp\left(\sum_i \int_0^t b_s^{(i)}(X_s)\,dX_s^{(i)} - \frac{1}{2}\int_0^t |b_s|^2 ds\right)$

& let $d\tilde{P}_T^x = Z_T\,dP^x$. $|b| \le C \Rightarrow \langle I(b(X), X)\rangle_T \le C^2 T \Rightarrow Z - 1 \in \mathcal{M}_c[0,T] \Rightarrow$ Girsanov applies.

Let $\tilde{W}_t = -\int_0^t b_s(X_s)\,ds + X_t$. Girsanov $\Rightarrow \{\tilde{W}_t\}_{t \le T}, \{\tilde{P}_T^x\}_{x \in \mathbb{R}^d}$ is a Brownian Family

& $X_t = X_0 + \int_0^t b_s(X_s)\,ds + \int_0^t d\tilde{W}_s$, & $X_0 = x$ P^x a.s. Now define $Q^\mu(A) = \int_{\mathbb{R}^d} \tilde{P}_T^x(A)\,d\mu(x)$.

Then $X_0 \sim \mu$ (under Q^μ) & is the desired weak solution.

QED

② Uniqueness. Say $X^{(i)}, W^{(i)}, P^{(i)}$ are two weak solutions with the same I.D.

Let $Z_t^{(i)} = \exp\left(-\int_0^t b_s(X_s^{(i)}) \cdot dW_s^{(i)} - \frac{1}{2}\int_0^t |b_s|^2 ds\right)$. As above, Girsanov applies.

Let $d\tilde{P}_T^{(i)} = Z_T^{(i)}\,dP^{(i)}$. By Girsanov, $\{X_t^{(i)}\}_{t \le T}$ is a BM under $\tilde{P}_T^{(i)}$, $\Rightarrow$ Laws of $X^{(i)}$ under $\tilde{P}_T^{(i)}$ are equal. Since $W_t^{(i)} = X_t^{(i)} - \int_0^t b_s(X_s^{(i)})\,ds$, $\Rightarrow$ Joint law's of $(X^{(i)}, W^{(i)})$ under $\tilde{P}_T^{(i)}$ are equal. $\Rightarrow$ laws of $\int_0^t b_s(X_s^{(i)}) \cdot dW_s^{(i)}$ under $\tilde{P}_T^{(i)}$ are equal. $\Rightarrow$ Joint law's of $(Z^{(i)}, X^{(i)})$ under $\tilde{P}_T^{(i)}$ are equal. $\Rightarrow \forall\, 0 \le t_1 \dots \le t_n \le T$, $A \in \mathcal{B}(\mathbb{R}^{nd})$, we have

$P((X_{t_1}^{(1)}, \dots X_{t_n}^{(1)}) \in A) = E\chi_{(\,) \in A} = \tilde{E}^{(1)} \chi_{(\,) \in A} \frac{1}{\tilde{Z}_T^{(1)}} = \tilde{E}^{(2)} \chi_{(X_{t_1}^{(2)} \dots X_{t_n}^{(2)}) \in A} \frac{1}{\tilde{Z}_T^{(2)}}$

$= P((X_{t_1}^{(2)} \dots X_{t_n}^{(2)}) \in A) \Rightarrow$ weak uniqueness.

QED.

Remark: ① Assumption $|b| \le C$ can be weakened to assuming Z is a martingale.

② Assumption $\sigma \equiv 1$ can be replaced with something that would guarantee strong existence of $dX = \sigma(X)\,dW$ & $Z_t = \exp\left(\int_0^t \sigma_s(X_s)^{-1} b_s(X_s) \cdot dW_s - \frac{1}{2}\int_0^t |\sigma_s^{-1}(X_s) b_s(X_s)|^2 ds\right)$ is a martingale.

(usually this requires an ellipticity assumption assuming $\sigma\sigma^* \ge \lambda I$).