Girsanov Thm.

Let $\tilde{W}_t = \int_0^t b_s\,ds + W_t$ (b adapted, meas & W a B.M.). Can we change the measure P to make $\tilde{W}$ a B.M.? Fix $T>0$. Say Z is a cont. martingale $\ni$ $Z>0$ a.s. & $EZ_T = 1$. Define $d\tilde{P}_T = Z_T\,dP$ & $\tilde{E}_T(\ ) = \int (\ )\,d\tilde{P}_T$.

Prop: $\tilde{M} \in \tilde{\mathcal{M}}_c[0,T]$ (i.e. M a mart wrt $\tilde{P}_T$) $\iff M = \tilde{M}Z \in \mathcal{M}_c[0,T]$.

Pf: Say $Z\tilde{M} \in \mathcal{M}_c$. Let $A \in \mathcal{F}_s$. $\int_A \tilde{M}_t\,d\tilde{P}_T = \int_A \tilde{M}_t Z_T\,dP = \int_A \tilde{M}_t Z_t\,dP = \int_A \tilde{M}_s Z_s\,dP = \int_A \tilde{M}_s\,d\tilde{P}_T$

$\Rightarrow \tilde{M} \in \tilde{\mathcal{M}}_c[0,T]$. Conversely ($\tilde{M} \in \tilde{\mathcal{M}}_c$) $\int_A M_t\,dP = \int_A \tilde{M}_t Z_t\,dP = \int_A \tilde{M}_t\,d\tilde{P}_T = \int_A \tilde{M}_s\,d\tilde{P}_T = \int_A \tilde{M}_s Z_T\,dP = \int_A \tilde{M}_s Z_s\,dP$ QED

Lemma: $\tilde{P}_T \ll P$ & ξ_n a seq of R.V. If $\lim P(|\xi_n - \xi| > \varepsilon) \to 0$, then $\lim \tilde{P}_T(|\xi_n - \xi| > \varepsilon) \to 0$.

(i.e. $(\xi_n) \to \xi$ in prob wrt $P \Rightarrow (\xi_n) \to \xi$ in prob wrt $\tilde{P}_T$).

Pf: Pick $\varepsilon_1 > 0$. $\exists \delta_1 > 0 \ni P(A) < \delta_1 \Rightarrow \tilde{P}_T(A) < \varepsilon_1$. Choose $N \ni n > N \Rightarrow P(|\xi_n - \xi| > \varepsilon) < \delta_1 \Rightarrow$ QED.

Cor: Let $\langle M \rangle^{\sim}$ = q.v. of M under $\tilde{P}_T$ & $\langle M \rangle$ = q.v. of M under P. Then $\langle M \rangle^{\sim} = \langle M \rangle$ $\tilde{P}_T$ a.s.

Pf: Recall $\langle M \rangle = \lim_{|\Delta_n| \to 0} \langle M \rangle^{\Delta_n}$ in probability, wrt P. $\overset{\text{Lemma}}{=} \lim_{|\Delta_n| \to 0} \langle M \rangle^{\Delta_n}$ in probab wrt $\tilde{P}_T$ QED.

$\therefore \langle \tilde{W} \rangle_t^{\tilde{P}} = \langle \tilde{W} \rangle_t^{P} = \begin{pmatrix} t & & \\ & t & \\ & & \ddots t \end{pmatrix}$ So to make $\tilde{W}$ a B.M. need to choose $Z \ni \tilde{W} \in \tilde{\mathcal{M}}_{c,loc}[0,T]$.

But $\tilde{W} \in \tilde{\mathcal{M}}_{c,loc} \iff Z\tilde{W} \in \mathcal{M}_{c,loc}$. But $d(Z\tilde{W}^{(i)})_t = Z_t b_t\,dt + Z_t\,dW_t^{(i)} + \tilde{W}^{(i)}\,dZ + d\langle Z, W^{(i)} \rangle$

$\therefore$ If $d\langle Z, W^{(i)} \rangle_t = -b_t^{(i)} Z_t\,dt$, we would be done. Enough to find $Z \ni$

$dZ_t = -\sum b_t^{(i)} Z_t\,dW_t^{(i)}$. Put $Z_t = \exp\left(-\sum \int_0^t b_s^{(i)}\,dW_s^{(i)} - \frac{1}{2}\int_0^t |b_s|^2\,ds\right)$

Let $Y_t^{(i)} = \int_0^t b_s^{(i)}\,dW_s^{(i)}$, $f(y,t) = \exp\left(-\sum y^{(i)} - \frac{1}{2}\int_0^t |b_s|^2\,ds\right)$

Then $Z_t = f(Y_t, t) \Rightarrow dZ_t = \partial_t f\,dt + \sum \partial_i f\,dY^{(i)} + \frac{1}{2}\sum \partial_{ij}^2 f\,d\langle Y^{(i)}, Y^{(j)} \rangle$

$$= -\tfrac{1}{2}|b_t|^2 Z_t\,dt - \sum Z_t b_t^{(i)}\,dW_t^{(i)} + \tfrac{1}{2} Z_t |b_t|^2\,dt = -\sum b_t^{(i)} Z_t\,dW_t^{(i)} \;!!$$

Thm (Cameron, Martin Girsanov) W a d-dim B.M., $b = \begin{pmatrix} b^{(1)} \\ \vdots \\ b^{(d)} \end{pmatrix}$ adapted, meas, $Z_t = \exp(\ )$.

Let $\tilde{W}_t^{(i)} = W_t^{(i)} + \int_0^t b_s^{(i)}\,ds$. If $Z - 1 \in \mathcal{M}_c[0,T]$ (not $\mathcal{M}_{c,loc}$), then $\{\tilde{W}_t \mid t \le T\}$ is a B.M. wrt $\tilde{P}_T$ (Pf: above).

Prop: Say $Z - 1 \in \mathcal{M}_c[0,T]$. Then $\tilde{M} \in \tilde{\mathcal{M}}_{c,loc}[0,T] \iff \exists M \in \mathcal{M}_{c,loc}[0,T] \ni \tilde{M}_t = M_t + \sum \int_0^t b_s^{(i)}\,d\langle M, W^{(i)} \rangle_s$.

Pf: Say first $M \in \mathcal{M}_{c,loc}[0,T]$ & $\tilde{M}_t = M_t + \sum \int_0^t b_s^{(i)}\,d\langle M, W^{(i)} \rangle_s$.

$d(Z\tilde{M})_t = Z_t\,d\tilde{M}_t + \tilde{M}_t\,dZ_t + d\langle Z, \tilde{M} \rangle = Z_t\,dM + \cancel{\sum Z_t b_t^{(i)}\,d\langle M, W^{(i)} \rangle_t} + \tilde{M}_t\,dZ_t - \cancel{\sum b_t^{(i)} Z_t\,d\langle M, W^{(i)} \rangle}$

$\Rightarrow Z\tilde{M} \in \mathcal{M}_{c,loc}[0,T]$. Conversely, say $\tilde{M} \in \tilde{\mathcal{M}}_{c,loc}[0,T]$. $\Rightarrow \tilde{M}Z \in \mathcal{M}_{c,loc}[0,T]$.

Itô $\Rightarrow$ the ratio of two semi-martingales is a loc semi-mg

$\therefore \tilde{M} = \frac{\tilde{M}Z}{Z}$ is a P loc. semimartingale $\Rightarrow \exists M \in \mathcal{M}_{c,loc}[0,T]$, & B, adapted, B.V. $\ni$

$\tilde{M} = M + B$. $\Rightarrow d(\tilde{M}Z) = \tilde{M}\,dZ + Z\,dM + Z\,dB + d\langle \tilde{M}, Z \rangle$. Since $\tilde{M}Z \in \mathcal{M}_{c,loc}[0,T]$ must have

$Z\,dB = -d\langle \tilde{M}, Z \rangle = -d\langle M, Z \rangle = \sum b_t^{(i)} Z_t\,d\langle M, W^{(i)} \rangle \Rightarrow B_t = \sum \int_0^t b_s^{(i)}\,d\langle M, W^{(i)} \rangle_s$. QED.

Prop: Let $\tilde{M} \in \mathcal{M}_{c,loc}[0,T]$ & $X \in \tilde{\mathcal{P}}^*(\tilde{M})$. Write $\tilde{M}_t = M_t + \sum \int_0^t b_s^{(i)}\,d\langle M, W^{(i)} \rangle_s$ for $M \in \mathcal{M}_{c,loc}[0,T]$.

Then $X \in \mathcal{P}^*(M)$ and $\underbrace{\int_0^T X\,d\tilde{M}}_{\tilde{P}_T \text{ Itô integral}} = \underbrace{\int_0^T X\,dM}_{P\text{- Itô integral.}} + \sum \int_0^T X_s b^{(i)}\,d\langle M, W^{(i)} \rangle$.

Pf: Note $Z_T > 0 \Rightarrow P \ll \tilde{P}_T$ $(\because \tilde{P}_T(A) = \int_A Z_T\,dP = 0 \Rightarrow P(A - \{Z_T = 0\}) = 0 \Rightarrow P(A) = 0)$

Also $\langle \tilde{M} \rangle^{\tilde{P}_T} = \langle \tilde{M} \rangle^P = \langle M \rangle$ $\therefore X \in \tilde{\mathcal{P}}^*(\tilde{M}) \Rightarrow \int_0^T |X_s|^2\,d\langle \tilde{M} \rangle_s < \infty$, $\tilde{P}$ a.s. $\Rightarrow \int_0^T |X_s|^2\,d\langle M \rangle_s < \infty$ P a.s.

$\Rightarrow X \in \mathcal{P}^*(M)$. Finally let $\tilde{N} = \int_0^T X\,dM + \sum \int_0^T X_s b^{(i)}\,d\langle M, W^{(i)} \rangle$. Then $\tilde{N} = \tilde{I}(X, \tilde{M}) \iff$

$\tilde{N} \in \tilde{\mathcal{M}}_{c,loc}[0,T]$ & $d\langle \tilde{N}, \tilde{\varphi} \rangle_t = X\,d\langle \tilde{M}, \tilde{\varphi} \rangle$ $\forall \varphi \in \tilde{\mathcal{M}}_{c,loc}[0,T]$.

Note, for $Y = I(X, M)$, $d\tilde{N} = dY + \sum b^{(i)}\,d\langle Y, W^{(i)} \rangle \overset{\text{By Prev prop.}}{\Rightarrow} \tilde{N} \in \tilde{\mathcal{M}}_{c,loc}[0,T]$. & since q.v. can be computed wrt P, $d\langle \tilde{N}, \tilde{\varphi} \rangle^{\tilde{P}_T} = X\,d\langle M, \tilde{\varphi} \rangle^P = X\,d\langle \tilde{M}, \tilde{\varphi} \rangle^P$ QED.

Recall: b adapted. W a B.M. $\tilde W_t = W_t + \int_0^t b_s\,ds$. Let $Z_t = \exp\left(-\sum_i \int_0^t b_s^{(i)}\,dW_s^{(i)} - \frac12 \int_0^t |b_s|^2\,ds\right)$

If $Z-1 \in \mathcal{M}_c[0,T]$, then $\{\tilde W_t \mid t\in[0,T]\}$ is a B.M. w.r.t. the new measure $\tilde P_T$ defined by $d\tilde P_T = Z_T\,dP$. [Remark: Itô $\Rightarrow Z-1 \in \mathcal{M}_{c,loc}[0,T]$. Need $Z \in \mathcal{M}_c[0,T]$]

Remark: $\forall T<\infty$, $\tilde P_T$ & P are equivalent. i.e. $\tilde P_T \ll P$ & $P \ll \tilde P_T$. For $T=\infty$, can't always find $\tilde P_\infty$! Sometimes possible (e.g. $\mathcal{F} = \mathcal{F}^W$ aug). However need not have $\tilde P_\infty \ll P$!

Prop: Let $\mu\in\mathbb{R}$, $\tau(b) = \inf\{t \mid W_t + \mu t = b\}$. Then $P(\tau(b)<\infty) = e^{\mu b - |\mu b|}$. [$\Rightarrow \tau(b)<\infty$ a.s. $\Leftrightarrow \mu b > 0$].

Setup Girsanov: $Z_t = \exp(\mu W_t - \frac12 \mu^2 t) \in \mathcal{M}_c$. Let $\mathcal{F}_t = \mathcal{F}_t^W$ (augmented) & $\forall t$, define $P_t^{(\mu)}(A) = \int_A Z_t\,dP$. Note, $A\in\mathcal{F}_s$ & $s\le t \Rightarrow P_s^{(\mu)}(A) = P_t^{(\mu)}(A)$. By Caratheodory, $\exists$ a measure $P^{(\mu)}$ on $\mathcal{F}_\infty \ni \forall A \in \mathcal{F}_t$, $P^{(\mu)}(A) = P_t^{(\mu)}(A)$. By Girsanov $\tilde W_t = W_t - \mu t$ is a BM under $P^{(\mu)}$.

$\Leftrightarrow W_t = \tilde W_t + \mu t$ is a BM with drift μ under $P^{(\mu)}$.

Lemma: Let τ be a stopping time with $P(\tau<\infty)=1$. Then $P^{(\mu)}(\tau<\infty) = E Z_\tau$.

Pf: $P^{(\mu)}(\tau\le t) = \int_{\{\tau\le t\}} Z_t\,dP = \int_{\{\tau\le t\}} E(Z_t \mid \mathcal{F}_{\tau\wedge t})\,dP = \int_{\{\tau\le t\}} Z_{\tau\wedge t}\,dP = \int_{\{\tau\le t\}} Z_\tau\,dP$.

$\therefore P^{(\mu)}(\tau<\infty) = \lim P^{(\mu)}(\tau\le n) = \lim \int_{\{\tau\le n\}} Z_\tau\,dP = E Z_\tau$ QED.

Pf of Prop: ① Let $\tau(b) = \inf\{t \mid W_t = b\}$. Know $P(\tau_b\in dt) = 2\frac{d}{dt}P(W_t > b)$ for $b>0$.

② Compute $P^{(\mu)}(\tau(b)<\infty) = E Z_{\tau(b)} = E e^{\mu b - \frac12 \mu^2 \tau(b)} = \dots = e^{\mu b - |\mu b|}$

But $W_t = b \Leftrightarrow \tilde W_t + \mu t = b$, $\therefore \tau(b) = t \Leftrightarrow \tilde\tau(b) = t$, where $\tilde\tau(b) = \inf\{t \mid \tilde W_t + \mu t = b\}$.

$\therefore P(\tau(b)<\infty) = P^{(\mu)}(\tilde\tau(b)<\infty) = P^{(\mu)}(\tau(b)<\infty) = E Z_{\tau(b)} = e^{\mu b - |\mu b|}$ QED

Remark: Say $\mu b<0$. Then $P^{(\mu)}(\tau(b)<\infty) = e^{-2|\mu b|} < 1$. But $P(\tau(b)<\infty)=1 \Rightarrow P^{(\mu)} \not\ll P$

Regularity of exponential martingales: $Z_t = \exp\left(-\sum \int_0^t b_s^{(i)}\,dW_s^{(i)} - \frac12 \int_0^t |b_s|^2\,ds\right)$

Let $M_t = -\sum \int_0^t b^{(i)}\,dW^{(i)} \in \mathcal{M}_{c,loc}$. Then $Z_t = \exp(M_t - \frac12\langle M\rangle_t)$.

Also, by Itô, $dZ = Z\,dM \Rightarrow Z-1 \in \mathcal{M}_{c,loc}$. Want conditions to guarantee $Z-1\in\mathcal{M}_c$

Remark: Z is always a super martingale. Consequently $EZ_t \le 1$ $\forall t$.

Pf: Let τ_n be a localizing seq for Z. $Z_t = \lim Z_{t\wedge\tau_n}$ & $Z\ge 0$. By Fatou,

$Z_s = \lim Z_{s\wedge\tau_n} = \lim E(Z_{t\wedge\tau_n} \mid \mathcal{F}_s) \overset{\text{Fatou}}{\ge} E(Z_t \mid \mathcal{F}_s)$ QED.

Prop: If $EZ_T = 1$, then $\{Z_t - 1\}_{t\le T} \in \mathcal{M}_c[0,T]$.

Pf: Let $\tau\le T$ be a stopping time. OST $\Rightarrow 1 = EZ_0 \ge EZ_\tau \overset{\tau\le t}{\ge} EZ_T = 1$

$\Rightarrow EZ_\tau = 1$ $\forall$ stopping times $\tau\le T$. $\Rightarrow Z-1 \in \mathcal{M}_c[0,T]$. QED

Prop: If $\langle M\rangle_T$ is bounded, then $Z-1 \in \mathcal{M}_c[0,T]$.

Pf: Claim 1: Say $\langle M\rangle_T \le C$ a.s. Then $P(\sup_{t\le T} M_t > \lambda) \le e^{-\lambda^2/2C}$

Pf: For $\theta\in\mathbb{R}$ define $Z_t(\theta) = \exp(\theta M_t - \frac12\theta^2\langle M\rangle_t)$. Know $Z(\theta)$ is always a super M.

$P(\sup_{t\le T} M_t > \lambda) \le P\left(\sup_{t\le T} \exp(\theta M_t - \frac12\theta^2\langle M\rangle_t) > \exp(\lambda\theta - \frac12\theta^2 C)\right)$

$= P\left(\inf_{t\le T} -Z_t(\theta) < -e^{\lambda\theta - \frac12\theta^2 C}\right) \le e^{-\lambda\theta + \frac12\theta^2 C}\left(E(-Z_T(\theta))^+ - E(-Z_0(\theta))\right) = e^{-\lambda\theta + \frac12\theta^2 C}$

Minimize in $\theta \Rightarrow \theta C = \lambda \Rightarrow P(\sup_{t\le T} M_t > \lambda) \le e^{-\lambda^2/C + \frac12 \lambda^2/C} = e^{-\lambda^2/2C}$ QED (Claim 1).

Claim 2: Let $M_t^* = \sup_{s\le t} M_t$. Then $\forall\theta\in\mathbb{R}$, $E\exp(\theta M_T^*) < \infty$.

Pf: $E e^{\theta M_T^*} = \int_0^\infty \theta e^{\lambda\theta} P(M^* > \lambda)\,d\lambda \le \int_0^\infty \theta e^{\lambda\theta - \lambda^2/2C}\,d\lambda < \infty$, $\forall\theta>0$. QED (Claim 2)

Pf of Prop: $E\sup_{t\le T} Z_t = E\sup_{t\le T}\exp(M_t - \frac12\langle M\rangle_t) \le E\exp(M_T^*) < \infty$. QED.

Cor: If $\sup_{t\le T}\sup_{\omega\in\Omega} |b_t(\omega)| < \infty$, & $Z_t = \exp\left(-\sum\int_0^t b_s^{(i)}\,dW_s^{(i)} - \frac12\int_0^t |b_s|^2\,ds\right)$ then $Z-1\in\mathcal{M}_c[0,T]$.

Thm: (Novikov) $E\exp(\frac12\langle M\rangle_T) < \infty$ & $Z_t = \exp(M_t - \frac12\langle M\rangle_t)$. Then $Z-1\in\mathcal{M}_c[0,T]$.

Thm: (Kazamaki) $E\exp(\frac12 M_t) < \infty$ $\forall t\le T \Rightarrow Z-1 \in \mathcal{M}_c[0,T]$.

Remark: $\langle M\rangle_T$ bdd $\Rightarrow$ Novikov applies $\Rightarrow$ Kazamaki applies. Pf: Let $Z_t = \exp(M_t - \frac12\langle M\rangle_t)$.

Then $Z_t^{1/2} = \exp(\frac12 M_t)\exp(-\frac12\langle M\rangle)^{1/2} \Rightarrow \exp(\frac12 M_t) = Z_t^{1/2}\exp(\frac12\langle M\rangle_t)^{1/2}$.

$\Rightarrow E\exp(\frac12 M_t) \le (EZ_t)^{1/2}\left(E\exp(\frac12\langle M\rangle_t)\right)^{1/2} \le \left(E\exp(\frac12\langle M\rangle_t)\right)^{1/2}$. QED.