Markov / Strong Markov Properties.

Def 1: An adapted process $\{X_t, \mathcal{F}_t\}$ is called a Markov process with i.d. μ if

① $X_0 \sim \mu$ & ② For $s<t$, $A \in \mathcal{B}(\mathbb{R}^d)$, $P(X_t \in A \mid \mathcal{F}_s) = P(X_t \in A \mid X_s)$

Def 2: A d-dimensional (time homog) Markov family is an adapted process $\{X_t, \mathcal{F}_t\}$ along with a family of measures $\{P^x\}_{x\in\mathbb{R}^d}$ s.t. ① $\forall F \in \mathcal{F}$, $x \to P^x(F)$ is universally meas, ② $P^x(X_0 = x) = 1$

③ $P^x(X_t \in A \mid \mathcal{F}_s) = P^x(X_t \in A \mid X_s)\ \forall s<t$ & ④ $P^x[X_{t+h}\in A \mid X_t = y] = P^y[X_h \in A]$ $P^x X_t^{-1}$ a.e. y

Recall: (Ω, ρ) a complete metric space. $\mathcal{U}(\Omega) = \bigcap_{\mu(\Omega)=1} \overline{\mathcal{B}(\Omega)}^{\mu}$ is the universal σ-alg on Ω & a function is said to be universally measurable. [Borel meas $\Rightarrow$ uni meas but not conv].

[Use univ meas because if $F \in$ augmentation of $\mathcal{F}$, $x \to P^x(F)$ need not be Borel]

Note: ④ $\Leftrightarrow$ $P^x[X_{t+h}\in A \mid X_t] = P^{X_t}(X_h \in A)$ P^x a.s.

Remark: ③ $\Leftrightarrow$ $E^x(f(X_t) \mid \mathcal{F}_s) = E^x(f(X_t)\mid X_s)$ $\forall f \in C_b(\mathbb{R}^d)$

and ④ $\Leftrightarrow$ $E^x(f(X_t) \mid X_s) = E^{X_s}(f(X_{t-s}))$ P^x a.s., $\forall$ bounded cts f.

Remark: Can equivalently replace ③ & ④ in the def of Markov by ⑤ $P^x(X_{t+h}\in A \mid \mathcal{F}_t) = P^{X_t}(X_h \in A)$

(Or equivalently $E^x(f(X_{t+h}) \mid \mathcal{F}_t) = E^{X_t} f(X_h)$ P^x a.s.)

Def: A d-dim Brownian family is a process $\{B_t, \mathcal{F}_t\}$ & a family of measures $\{P^x\}_{x\in\mathbb{R}^d}$ s.t.

① $\forall F \in \mathcal{F}$, $x \to P^x(F)$ is universally meas. ② $\forall x$, $B_0 = x$, P^x a.s.

& ③ $\forall x$, B is a d-dim B.M. (under P^x).

Existence: Let $\Omega = [C[0,\infty)]^d$ & P a probability measure on Ω s.t. the process $W_t(\omega) = \omega(t)$ is a B.M.

Define $P^x(F) = P(F - x) = P\{\omega \in \Omega \mid \omega(\cdot) + x \in F\}$. Then $\{W, \mathcal{F}_t\}$ & $\{P^x\}$ is a B. fam.

Prop: B.M. is a Markov Process & a Brownian family is a Markov family.

Pf: Let $s<T$, $f \in C_b^2(\mathbb{R}^d)$, & let $\varphi_s(x) = E^x f(W_{T-s}) = f * G_{T-s}(x)$.

Trick from last time $\Rightarrow$ $\varphi \in C^{1,2}((-\infty,T]\times\mathbb{R}^d)$, $\partial_s \varphi + \frac{1}{2}\Delta\varphi = 0$ & $\varphi_T = f$.

BV terms cancel.

Then $f(W_T) = \varphi_T(W_T) = \varphi_s(W_s) + \int_s^T \partial_i \varphi_r(W_r)\, dW_r \Rightarrow E^x(f(W_T) \mid \mathcal{F}_s) = \varphi_s(W_s) = E^{W_s} f(W_{T-s})$. P^x a.s.

Now for $f \in C_b(\mathbb{R}^d)$, let $(f^{(n)}) \xrightarrow{\text{ptwise}} f$, $f^{(n)} \in C_b^2$, $\|f^{(n)}\|_\infty \le \|f\|_\infty$. Then

$E^x(f(W_T) \mid \mathcal{F}_s) = \lim E^x(f^{(n)}(W_T) \mid \mathcal{F}_s) = \lim E^{W_T} f^{(n)}(W_{T-s}) = E^{W_T} f(W_{T-s})$ P^x a.s. QED

Def: A prog. meas process $\{X_t, \mathcal{F}_t\}_{t\ge0}$ is a strong Markov process if $\forall$ optional times τ, & $\forall h \in \mathbb{R}$

$P(X_{\tau+h}\in A \mid \mathcal{F}_{\tau+}) = P(X_{\tau+h} \mid X_\tau)$. ($\Leftrightarrow$ $E(f(X_{\tau+h}) \mid \mathcal{F}_{\tau+}) = E(f(X_{\tau+h}) \mid X_\tau)$ $\forall f \in L^\infty(\mathbb{R}^d, \mathcal{B})$)

Def: A strong Markov family is a progressively meas process $\{X_t, \mathcal{F}_t\}$ ($\mathcal{F}_t$ satisfies usual cond.) and a family of measures $\{P^x\}_{x\in\mathbb{R}^d}$ s.t. ① $\forall F \in \mathcal{F}$, $x \to P^x(F)$ is universally meas,

② $X_0 = x$, P^x a.s., ③ $P^x(X_{\tau+h}\in A \mid \mathcal{F}_{\tau+}) = P^x(X_{\tau+h}\in A \mid X_\tau)$ P^x a.s. on $\{\tau<\infty\}$.

& ④ $P^x(X_{\tau+h}\in A \mid X_\tau) = P^{X_\tau}(X_h \in A)$ P_x a.s. on $\{\tau<\infty\}$. ($\Leftrightarrow$ $P^x(X_{\tau+h}\in A \mid X_\tau = y) = P^y(X_h \in A)$ $P^x X_\tau^{-1}$ a.e. y)

Remark: As before ③ & ④ can be replaced with ⑤ $P^x(X_{\tau+h}\in A \mid \mathcal{F}_{\tau+}) = P^{X_\tau}(X_h \in A)$ P^x a.s.

or alternately "$E^x(f(X_{\tau+h}) \mid \mathcal{F}_{\tau+}) = E^{X_\tau}(f(X_h))$ P^x a.s. on $\{\tau<\infty\}$ $\forall f \in L^\infty(\mathbb{R}^d, \mathcal{B})$"

Remark: ~~As before~~, ⑤ $\Leftrightarrow$ ⑤': $\forall F \in \mathcal{B}(C[0,\infty)^d)$, $P^x(X_{\tau+\cdot} \in F \mid \mathcal{F}_{\tau+}) = P^{X_\tau}(X \in F)$ P^x a.s. on $\{\tau<\infty\}$.

Prop: B.M. is a S. Markov process & a Brownian Fam is a S. Markov fam.

Lemma: Let $\tau < \infty$ a.s. & $\{B_t, \mathcal{F}_t\}$ a B.M. Let $W_t = B_{\tau+t} - B_\tau$. Then W is a B.M. & ind of $\mathcal{F}_\tau$.

Pf: Let $\mathcal{G}_t = \mathcal{F}_{\tau+t}$. Then $E(W_t \mid \mathcal{G}_s) = E(B_{\tau+t} - B_\tau \mid \mathcal{F}_{\tau+s}) = B_{\tau+s} - B_\tau = W_s$. $\Rightarrow$ $\{W_t, \mathcal{G}_t\} \in \mathcal{M}_c$.

OST Doesn't apply

Also, $\langle W \rangle_t = \tau + t - \tau = t$. (Note: $\langle X-Y\rangle \ne \langle X\rangle - \langle Y\rangle$ in general). Levy $\Rightarrow$ W a B.M.

Also $W_0 = 0$ so $W_t = W_t - W_0$ is ind of $\mathcal{G}_0 = \mathcal{F}_\tau$. QED.

Pf of S. Markov: As before $\varphi_s(x) = E^x f(W_{T-s})$. Let $B_s = W_{\tau+s}$, $\mathcal{G}_s = \mathcal{F}_{\tau+s}$. Then $\{B_s, \mathcal{G}_s\}$ is a B.M. with $B_0 = W_\tau$ a.s. (Some specified initial law). Assume first $f \in C_b^2$.

Hence $\varphi \in C^{2,1}$. By Itô, $f(W_{\tau+T}) = \varphi_T(B_T) = \varphi_0(B_0) + \int_0^T \cancelto{0}{(\partial_s \varphi_s(B_s) + \frac{1}{2}\Delta\varphi_s(B_s))}\, ds +$

$+ \int_0^T \sum \partial_i \varphi_s(B_s)\, dB_s^{(i)} \Rightarrow E^x(f(W_{\tau+T}) \mid \mathcal{F}_{\tau+}) = E^x(\varphi_0(B_0) + \sum \int_0^T (\cdot)\, dB^{(i)} \mid \mathcal{G}_0) = \varphi_0(W_\tau) = E^{W_\tau} f(W_T)$

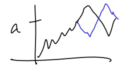
P^x a.s. QED.

Passage times: Let $a \in \mathbb{R}$, & $\tau_a = \tau(a) = \inf\{t \ge 0 \mid B_t = a\}$ = exit time from $(-\infty, a)$.

$\mapsto 1 - \operatorname{erf}(a/\sqrt{2t})$ $\quad \approx a/\sqrt{t}$ for large t

Claim: $P^0(\tau_a < t) = 2P^0(B_t > a)$. (Note $\Rightarrow \tau_a < \infty$ a.s. Also, $E\tau^a = \int_0^\infty P(\tau_a > t)\,dt = \int_0^\infty P^0(|B_t| < a)\,dt = \infty$)

Lemma: (André's Reflection Principle) τ a stopping time. Then $\tilde B_t = B_{\tau\wedge t} - (B_t - B_{\tau\wedge t})$ is a B.M.

Intuition: $\tilde B_t = \begin{cases} B_t & t \le \tau \\ B_\tau - (B_t - B_\tau) & t > \tau \end{cases}$ Reflected about B_τ. For $\tau = \tau_a$,  ← Reflected after hitting a.

Pf: ① Picture $\Rightarrow \langle \tilde B \rangle_t = \langle B \rangle_t = t$. OST $\Rightarrow E(\tilde B_t \mid \mathcal{F}_s) = B_{\tau\wedge s} - (B_s - B_{\tau\wedge s}) = \tilde B_s$.

$\therefore \tilde B \in \mathcal{M}_c$ & by Levy, must be a B.M.

Pf of Claim: Let $\tilde B_t = B_{\tau\wedge t} - (B_t - B_{\tau\wedge t})$ & $\tilde\tau_a = \inf\{t \ge 0 \mid \tilde B_t = a\}$. Observe $\tilde\tau_a = \tau_a$ a.s.

Then $P^0(\tau_a < t) = P^0(\tau_a < t \,\&\, B_t > a) + P^0(\tau_a < t \,\&\, B_t \le a)$.

Note $\{\tau_a < t\} \subseteq \{B_t > a\}$ by continuity of paths. Also, $B_t \le a \Rightarrow \tilde B_t \ge a$.

$\therefore P^0(\tau_a < t) = P^0(B_t > a) + P^0(\tilde\tau_a < t \,\&\, \tilde B_t > a) = P^0(B_t > a) + P^0(\tilde B_t > a) = 2P(B_t > a)$ QED.

0–1 Laws: Let $\mathcal{F}_t = \mathcal{F}_t^B$ augmented. ① Blumenthal: $A \in \mathcal{F}_{0^+} \Rightarrow P(A) = 0$ or 1 (more gen $\mathcal{F}_{t^+} = \mathcal{F}_t$)

② Kolmogorov: $\mathcal{G}_s = \sigma\left(\bigcup_{t \ge s} \sigma(B_t)\right)$ & $\mathcal{G}_{\infty^-} = \bigcap_{s \ge 0} \mathcal{G}_s$. $A \in \mathcal{G}_{\infty^-} \Rightarrow P(A) = 0$ or 1.

Pf: **Claim:** $\forall f_1, \dots, f_n \in L^\infty(\mathbb{R}^d, \mathcal{B})$, $0 \le t_1, \dots, t_n$, $E(f_1(B_{t_1}) \cdots f_n(B_{t_n}) \mid \mathcal{F}_{0^+}) = E(f_1(B_{t_1}) \cdots f_n(B_{t_n}) \mid B_0)$

Pf: ① $n=1$: S.Markov $\Rightarrow E(f(B_{t_1}) \mid \mathcal{F}_{0^+}) = E(f(B_1) \mid B_0)$

② $n=2$: $E(f_1(B_{t_1}) f_2(B_{t_2}) \mid \mathcal{F}_{0^+}) = E(f_1(B_{t_1}) E(f_2(B_{t_2}) \mid \mathcal{F}_{t_1^+}) \mid \mathcal{F}_{0^+})$

$= E(f_1(B_{t_1}) E(f_2(B_{t_2}) \mid B_{t_1}) \mid \mathcal{F}_{0^+}) = E(\text{—} \mid B_0) = E(f_1(B_{t_1}) f_2(B_{t_2}) \mid B_0)$

Claim follows by induction.

$\forall A \in \mathcal{F}_t^B$, $\chi_A = \lim \sum_{i=1}^m \prod_{j=1}^n f_{ij}(B_{t_{ij}})$. $\therefore E(\chi_A \mid \mathcal{F}_{0^+}) = E(\chi_A \mid B_0) = E\chi_A$.

If $A \in \mathcal{F}_{0^+}$, $E(\chi_A \mid \mathcal{F}_{0^+}) = \chi_A$. $\Rightarrow$ $\chi_A = E\chi_A$ a.s. $\Rightarrow P(A) = 0$ or 1. $\Rightarrow$ Blumenthal.

Rmk: Can similarly show that $\mathcal{F}_{t^+} = \mathcal{F}_t$ for any S. Markov Process.

Pf of Kolmogorov: Let $W_t = tB_{1/t}$. Then $\mathcal{G}_t = \sigma\left(\bigcup_{s \ge t} \sigma(B_s)\right) = \sigma\left(\bigcup_{s \le t} \sigma(W_s)\right) = \mathcal{F}_t^W$.

$\Rightarrow \mathcal{G}_{\infty^-} = \mathcal{F}_{0^+}^W$. Blumenthal $\Rightarrow \forall A \in \mathcal{F}_{0^+}^W$, $P(A) = 0$ or 1 $\Rightarrow$ QED.

Prop: Let $\sigma_+ = \inf\{t > 0 \mid B_t > 0\}$, $\sigma_- = \inf\{t > 0 \mid B_t < 0\}$, $\sigma = \inf\{t > 0 \mid B_t = 0\}$. Then $\sigma_+ = \sigma_- = \sigma \overset{a.s.}{=} 0$.

Pf: $\{\sigma_+ = 0\} = \bigcap_{n=1}^\infty \{\sigma_+ \le 1/n\} \in \bigcap_1^\infty \mathcal{F}_{1/n} \subseteq \mathcal{F}_{0^+}$. $\Rightarrow P\{\sigma_+ = 0\} \in \{0, 1\}$ (Blumenthal)

But $\{\sigma^+ \le t\} \supseteq \{B_t > 0\} \Rightarrow P(\sigma^+ \le t) \ge 1/2\ \forall t$.

$\Rightarrow P(\sigma_+ = 0) = \lim P(\sigma_+ \le 1/n) \ge 1/2$. $\Rightarrow P(\sigma_+ = 0) = 1$. $\overset{\text{by}}{|||}$ $P(\sigma_- = 0) = 1$.

B has const sign on $(0, \sigma)$. $\Rightarrow \{\sigma > 0\} \subseteq \{\sigma_+ > 0\} \cup \{\sigma_- > 0\} \Rightarrow P(\sigma > 0) = 0$

Cor: Let τ = any finite stopping time. $\sigma_+ = \inf\{t > \tau \mid B_t > B_\tau\}$, $\sigma_- = \inf\{t > \tau \mid B_t < B_\tau\}$, $\sigma = \inf\{t > \tau \mid B_t = B_\tau\}$.

Then $\sigma_+ = \sigma_- = \sigma = \tau$ a.s. (Eg $\tau = \tau_a$. $\inf\{t > \tau_a \mid B_t = a\} = \tau_a$ a.s.)

Pf: Let $W_t = B_{\tau+t} - B_\tau$. Then $\sigma^+ - \tau = \inf\{t > 0 \mid W_t > 0\}$ is an $\mathcal{F}_t^W$-stopping time.

S.Markov $\Rightarrow$ W a B.M. Prev prop $\Rightarrow \sigma^+ - \tau = 0$ a.s. QED

Cor: B.M. is not monotone on any interval (a.s.)

Cor: Let $\tau = \sup\{t \le 1 \mid W_t = 0\}$. Then τ is not a stopping time.

Pf: Say τ is a stopping time. Then $\inf\{t > \tau \mid W_t = 0\} = \tau$ a.s. by above.

But $W_t \ne 0\ \forall t \in (\tau, 1)$. $\Rightarrow \tau \ge 1$ a.s. $\Rightarrow \tau = 1$ a.s. $\Rightarrow W_1 = 0$ a.s. $\Rightarrow\Leftarrow$ QED.

Prop: $C = C(\omega) = \{t \ge 0 \mid B_t(\omega) = 0\}$. Then P^0 a.s., C is closed, unbdd above, Lebesgue measure 0 & has no isolated points.

Pf: ① Measure 0: $\lambda(C) = \int_0^\infty \chi_C(t)\,dt \Rightarrow E\lambda(C) = E\int_0^\infty \chi_C(t)\,dt = \int_0^\infty E\chi_C(t)\,dt = \int_0^\infty P(B_t = 0)\,dt = 0$.

$\Rightarrow \lambda(C) = 0$ a.s.

② Isolated points: C has an isolated pt $\Leftrightarrow \exists s \in \mathbb{Q} \ni$ for $\tau_s = \inf\{t \ge s \mid B_t = 0\}$

& $\tau_s' = \inf\{t > \tau_s \mid B_t = 0\}$ & $P(\tau_s' > \tau_s) > 0$. $\Rightarrow$ no isolated points.

③ Also, $0 \in C \Rightarrow 0$ a limit point. Time inversion $\Rightarrow \infty$ is a limit pt $\Rightarrow$ un bdd. QED

Running Maximum. Let $M_t = \sup_{s\le t} B_s$. Goal: Compute $P(B_t\in da, M_t\in db)$.

Prop: Let $X = \{X_t, \mathcal{F}_t\}, \{P^x\}_{x\in\mathbb{R}^d}$ be a R.C. strong Markov family. Let τ be any optional time & h be any $\mathcal{F}_{\tau+}$ meas. random time. Then $\forall$ bdd Borel f, $\forall x\in\mathbb{R}^d$,

$E^x[f(X_{\tau+h}) \mid \mathcal{F}_{\tau+}] = \left(E^{X_\tau} f(X_s)\right)_{s=h}$ P^x a.s. on $\{\tau+h<\infty\}$ [i.e. $\forall s\in\mathbb{R}$, define the operator U_s by $U_s f(x) = E^x f(X_s)$. Then $E^x[f(X_{\tau+h})\mid\mathcal{F}_{\tau+}] = U_h f(X_\tau)$]

Pf: ① Say $h = \sum_1^\infty h_i\chi_{A_i}$, where $A_i\in\mathcal{F}_{\tau+}$, disj. Then $E^x(f(X_{\tau+h})\mid\mathcal{F}_{\tau+}) = E^x(\sum\chi_{A_i} f(X_{\tau+h_i})\mid\mathcal{F}_{\tau+})$

$= \sum\chi_{A_i} E^x(f(X_{\tau+h_i})\mid\mathcal{F}_{\tau+}) = \sum\chi_{A_i} U_{h_i} f(X_\tau) = U_h f(X_\tau)$. So theorem is true for when h has countable range. Let (h_n) be a sequence of $\mathcal{F}_{\tau+}$ meas random times $\ni (h_n)\searrow h$. Then for $f\in C_b(\mathbb{R}^d)$, $f(X_{\tau+h_n}) \xrightarrow{\text{a.s. \& dominated}} f(X_{\tau+h})$ by R.C. of X. QED.

Running Maximum. Let $M_t = \sup_{s\le t} B_s$. Then for $t>0$, $a\le b$, $b\ge 0$ we have

$$P^0(B_t\in da, M_t\in db) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} e^{-\frac{(2b-a)^2}{2t}}\, da\, db.$$

passage time to b.

Pf: $\forall t$, $P^b(B_t\le a) = P^b(B_t\ge b+(b-a))$ by symmetry. So now on $\{\tau(b)\le t\}$.

$P^0(B_t\le a\mid\mathcal{F}_{\tau(b)+}) = \left[P^b(B_s\le a)\right]_{s=t-\tau(b)} = \left[P^b(B_s\ge 2b-a)\right]_{s=t-\tau(b)}$

$= P^0(B_t\ge 2b-a\mid\mathcal{F}_{\tau(b)+})$.

$\therefore P^0(B_t\le a, M_t\ge b) = P^0(B_t\le a, \tau_b\le t) = \int_{\{\tau_b\le t\}} P(B_t\le a\mid\mathcal{F}_{\tau(b)}) = \int_{\{\tau_b\le t\}} P(B_t\ge 2b-a\mid\mathcal{F}_{\tau(b)+})$

$= P(B_t\ge 2b-a, \tau_b\le t) = P(B_t\ge 2b-a) = \frac{1}{\sqrt{2\pi t}}\int_{2b-a}^\infty e^{-x^2/2t}dx$ & differentiate QED

Remark: Consider now the process $X_t = |B_t|$ (Reflected BM) & $Y_t = M_t - B_t$. Claim!! Both X & Y are Markov processes & have the same finite dimensional distributions! In fact the transition densities (of both processes) are given by

$$P(X_{t+h}\in dx\mid X_t = y) = p(h,x,y) + p(h,x,-y) \quad\text{where } p(s,x,y) = \frac{1}{\sqrt{2\pi s}} e^{-(x-y)^2/2s}$$

Laws of Iterated Logarithm: Let W be a standard 1D B.M. The foll hold a.s.:

① $\overline{\lim_{t\to0^+}} \frac{W_t}{\sqrt{2t\ln\ln(1/t)}} = 1$ ② $\underline{\lim_{t\to0^+}} \frac{W_t}{\sqrt{2t\ln\ln(1/t)}} = -1$

③ $\overline{\lim_{t\to\infty}} \frac{W_t}{\sqrt{2t\ln\ln t}} = 1$ & $\underline{\lim_{t\to\infty}} \frac{W_t}{\sqrt{2t\ln\ln(t)}} = -1$

Pf: Replacing W_t with $-W_t$, ① ⇒ ②. Replacing W_t with $tW_{1/t}$, ① ⇒ 3 & ② ⇒ ④.

So it suffices to prove ①. Note, Itô ⇒ $X_t = e^{\lambda W_t - \frac12\lambda^2 t}$ is a martingale ($\forall\lambda$). So

$P\left(\sup_{s\le t}(W_s - \tfrac{\lambda}{2}s)\ge\beta\right) = P\left(\sup_{s\le t} e^{\lambda W_s - \frac{\lambda^2}{2}s}\ge e^{\lambda\beta}\right) \le (E X_t^+)e^{-\lambda\beta}\le e^{-\lambda\beta}$.

Let $h(t) = \sqrt{2t\ln\ln(1/t)}$. Pick $\delta,\theta\in(0,1)$ [$\delta\to0$, $\theta\to1$].

Choose $\lambda = \frac{(1+\delta)h(\theta^n)}{\theta^n}$, $\beta = \frac{h(\theta^n)}{2}$, $t=\theta^n$. Then $P\left(\sup_{s\le\theta^n}(W_s-\tfrac{\lambda}{2}s)\ge\beta\right)\le e^{-\lambda\beta}$

$= \exp\left(-\frac{(1+\delta)h(\theta^n)^2}{2\theta^n}\right) = \exp\left(-\frac{(1+\delta)2\theta^n\ln\ln(\theta^{-n})}{2\theta^n}\right) = (\ln\theta^{-n})^{-(1+\delta)} = \left(\frac{1}{n\ln(\frac1\theta)}\right)^{1+\delta}$

Since $\sum\frac{1}{(n\ln(\frac1\theta))^{1+\delta}}<\infty$, By Borel Cantelli, $\exists\,\Omega_{\theta,\delta}\subseteq\Omega \ni P(\Omega_{\theta,\delta})=1$

and $\forall\omega\in\Omega_{\theta,\delta}$, $\exists N(\omega)\ni\forall n>N(\omega)$, $\sup_{s\le\theta^n}\left(W_s - \frac{(1+\delta)}{2\theta^n}h(\theta^n)s\right) < \frac{h(\theta^n)}{2}$

⇒ $\forall n>N(\omega)$, $\theta^{n+1}\le t<\theta^n$, $W_t\le\sup_{\theta^{n+1}\le s\le\theta^n} W_s \le \frac{h(\theta^n)}{2} + \inf_{\theta^{n+1}\le s\le\theta^n}\frac{(1+\delta)}{2\theta^n}h(\theta^n)s$

$\le\frac{h(\theta^n)}{2} + \frac{(1+\delta)}{2}h(\theta^n)\theta \le h(\theta^n)(1+\frac\delta2) = \sqrt{2\theta^n\ln\ln(\frac{1}{\theta^n})}\,(1+\frac\delta2)$

$\le(1+\frac\delta2)\sqrt{2t\ln\ln(\frac1t)}\cdot\left(\frac{\theta^n}{t}\right)^{1/2}\le(1+\frac\delta2)h(t)\,\theta^{-1/2}$

$\therefore \overline{\lim_{t\to0^+}}\frac{W_t}{h(t)}\le(1+\frac\delta2)\theta^{-1/2}$ a.s. on $\Omega_{\theta,\delta}$. Let $\theta\to1$, $\delta\to0$ along rationals & get $\overline{\lim_{t\to0^+}}\frac{W_t}{h(t)}\le1$ a.s. on Ω.

Opposite direction: Let $A_n = \{W_{\delta^n} - W_{\delta^{n+1}}\ge\sqrt{1-\delta}\,h(\delta^n)\}$ for $\delta\in(0,1)$. ($\delta\to0$)

$P(A_n) = P\left(\frac{W_{\theta^n}-W_{\theta^{n+1}}}{(\theta^n-\theta^{n+1})^{1/2}}\ge\sqrt{2\ln\ln(\frac1\theta)}\right) = P\left(N(0,1)\ge\sqrt{2\ln\ln\frac{1}{\theta^n}}\right)$ ... suffering ... $\ge\frac{c}{n\sqrt{\ln n}}$

⇒ $\sum P(A_n)$ diverges (& the A_n's are independent). Borel Cantelli ⇒ $\exists\,\Omega_\delta\ni\forall\omega\in\Omega_\delta$,

$\forall k\ \exists n = n(k,\omega)>k\ni\omega\in A_n$. Note by the first half, $\overline{\lim}\ -\frac{W_t}{h(t)}\le1$ a.s.

& so, for large n, $-W_{\delta^{n+1}}\le2h(\delta^{n+1}) = 2\sqrt{2\delta^{n+1}\ln\ln\frac{1}{\delta^{n+1}}}\le4\sqrt\delta\,h(\delta^n)$

$\therefore$ for large n, $W_{\delta^n}\ge W_{\delta^{n+1}} + \sqrt{1-\delta}\,h(\delta^n)\ge(\sqrt{1-\delta}-4\sqrt\delta)\,h(\delta^n)$. Sending $n\to\infty$,

get $\overline{\lim_{t\to0^+}}\frac{W_t}{h(t)}\ge\sqrt{1-\delta}-4\sqrt\delta$ a.s. on Ω_δ. Let $\delta\to0$ along rationals ⇒ QED