Brownian Motion

Let $\xi_1, \xi_2, \dots$ be i.i.d, mean 0, variance 1. $\xi_0 = 0$. $k \in \mathbb{N}$.

Define $B^{(1)}_t = \sum_1^\infty \xi_n (t\wedge(n+1) - t\wedge n)$. $B^{(k)} = \sum \frac{\xi_n}{\sqrt{k}}\left(t\wedge\left(\frac{n+1}{k}\right) - t\wedge\left(\frac{n}{k}\right)\right)$

Send $k \to +\infty$. $B^{(k)}_t - B^{(k)}_s \propto k(t-s)$ steps of a R.W. variance $\frac{1}{k}$. CLT $\Rightarrow B^{(k)}_t - B^{(k)}_s \sim N(0, t-s)$

Also, CLT $\Rightarrow (B^{(k)}_s, B^{(k)}_t - B^{(k)}_s) \longrightarrow N\left(0, \begin{pmatrix} s & 0 \\ 0 & t-s\end{pmatrix}\right)$. N.C. $\Rightarrow B_t - B_s$ ind of B_s.

Def: A standard 1D B.M. is an adapted process $\{B_t, \mathcal{F}_t\} \ni$ ① $B_0 = 0$ a.s.

② $B_t - B_s \sim N(0, t-s)$ & is independent of $\mathcal{F}_s$ ③ B has cts trajectories.

Note: $B \in M^2_c$. $[\because$ ① $\forall t$, $EB_t^2 = t < \infty$ & $E(B_t \mid \mathcal{F}_s) = E(B_s + B_t - B_s \mid \mathcal{F}_s) = B_s]$

Construction of BM. Let $R^{[0,\infty)} = \{f \mid f : [0,\infty) \to R\} = \Omega$. An n-dimensional cylinder set is a set of the form $\{\omega \mid (\omega(t_1), \dots \omega(t_n)) \in A\}$ for some $A \in \mathcal{B}(R^n)$.

Let $\mathcal{C} = \{$all cyl. sets$\}$. Note $\mathcal{B}(R^{[0,\infty)}) = \sigma(\mathcal{C})$. Let $X_t(\omega) = \omega(t)$ be the canonical process.

Def: Let $T = \{(t_1, \dots t_n) \mid n \in \mathbb{N}, t_i \geq 0, t_i \neq t_j \text{ for } i \neq j\}$. Say $\forall t = (t_1 \dots t_n) \in T$, we have a probability measure Q_t on $(R^n, \mathcal{B}(R^n))$. Then $\{Q_t\}_{t\in T}$ (the fam of f.d. dist) is said to be consistent if ① If $s = (t_1 \dots t_n)$ & $t = (t_{\sigma(1)} \dots t_{\sigma(n)})$, $\sigma \in S_n$,

Intuition: $Q_t(A) = P(X_t \in A)$

Then $Q_s(A_1 \times \dots A_n) = Q_t(A_{\sigma_1} \times \dots A_{\sigma_n})$. and ② $s = (t_1, \dots t_{n-1})$, $t = (t_1, \dots t_n)$

then $Q_s(A_1 \times \dots A_{n-1}) = Q_t(A_1 \times \dots A_{n-1} \times R)$.

Remark: Let P be a prob measure on $(R^{[0,\infty)}, \mathcal{B}(R^{[0,\infty)}))$. Let $t = (t_1 \dots t_n)$ & set

(*) $Q_t(A_1 \times \dots A_n) = P\{\omega \in R^{[0,\infty)} \mid \omega(t_i) \in A_i \ \forall i \in \{1 \dots n\}\}$. Then $\{Q_t\}_{t\in T}$ is consistent.

Thm: (Kolmogorov's consistency thm) (proved by Daniell) If $\{Q_t\}_{t\in T}$ is a consistent family of f.d. dist, then $\exists$ a prob. measure on $\mathcal{B}(R^{[0,\infty)}) \ni$ (*) holds. (Hence the coordinate mapping process is a stochastic process with the desired distribution)

Cor: $\exists$ a process $\{B_t\} \ni$ ① $B_0 = 0$ a.s. ② $0 \leq t_1 < \dots t_n \Rightarrow (B_{t_1}, B_{t_2} - B_{t_1}, \dots B_{t_n} - B_{t_{n-1}}) \sim N\left(0, \begin{pmatrix} t_1 & & \\ & t_2 - t_1 & \\ & & \ddots \end{pmatrix}\right)$

(N.C. $\Rightarrow$ Ind inc).

Pf: Let $\Omega = R^{[0,\infty)}$, $B_t(\omega) = \omega(t)$. $\forall 0 \leq t_1 < t_2 \dots t_m \in R$, $A \in \mathcal{B}(R^m)$

Let $Q_{(t_1, \dots t_m)}(A) = \int_A G_{t_1}(x_1) G_{t_2 - t_1}(x_2 - x_1) \cdots G_{t_m - t_{m-1}}(x_m - x_{m-1})\, dx_1 \dots dx_m$; $G_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$

Unfortunately $C[0,\infty) \notin \mathcal{B}(R^{[0,\infty)})$ so can't get continuity easily

Consistency is immediate $\therefore$ Kolmogorov $\Rightarrow \exists$ a measure P on $(R^{[0,\infty)}, \mathcal{B}(R^{[0,\infty)}))$ with f.d. distributions Q. Independence follows from def of Q. QED

Thm: (Kolmogorov Čentsov): Say $\{X_t\}_{t\in T}$ is a process $\ni E|X_t - X_s|^\alpha \leq C|t-s|^{1+\beta}$, $\alpha, \beta > 0$

Doesn't need adapted.

Then X has a continuous modification. Further $\forall \gamma < \beta/\alpha$, $\exists \delta > 0$ and a +ve random variable $h \ni P\left[\sup_{0 \leq |t-s| \leq h(\omega)} \frac{|X_t(\omega) - X_s(\omega)|}{|t-s|^\gamma} \leq \delta\right] = 1$.

Pf: Say $T = 1$, $c = 1$. $\forall \varepsilon > 0$, $P(|X_t - X_s| > \varepsilon) \leq \frac{1}{\varepsilon^\alpha} E|X_t - X_s|^\alpha \leq \frac{|t-s|^{1+\beta}}{\varepsilon^\alpha} \xrightarrow[t\to s]{} 0$

$\Rightarrow \lim_{t\to s} X_t = X_s$ (in probability). Let $s = \frac{k-1}{2^n}$, $t = \frac{k}{2^n}$, $\varepsilon = 2^{-\gamma n}$.

$\Rightarrow P\left(\max_{k \leq 2^n} |X_{k/2^n} - X_{\frac{k-1}{2^n}}| > 2^{-\gamma n}\right) = P\left(\bigcup_{k \leq 2^n} |X_{k/2^n} - X_{\frac{k-1}{2^n}}| > 2^{-\gamma n}\right)$

$k/2^n$ · s t

$\leq \sum_{k=1}^{2^n} P\left(|X_{k/2^n} - X_{\frac{k-1}{2^n}}| > 2^{-\gamma n}\right) \leq \sum_{k=1}^{2^n} 2^{-n(1+\beta)} / 2^{-\alpha\gamma n} = 2^{-n(\beta - \alpha\gamma)}$

By Borel Cantelli, $\exists \Omega^* \subseteq \Omega$ with $P(\Omega^*) = 1 \ni \forall \omega \in \Omega^*$ $\exists n_0(\omega) \ni$

Ω^* doesn't happen i.o.

$\forall n > n_0(\omega)$, $\max_{k \leq 2^n} |X_{k/2^n}(\omega) - X_{\frac{k-1}{2^n}}(\omega)| \leq 2^{-\gamma n}$. (like Hölder cont on diadic rationals).

Let $D_n = \{k/2^n, k \leq 2^n\}$ & $D = \bigcup D_n$. **Claim ①:** For $n_0(\omega) \leq m < n$, $s, t \in D_n$, with

$t = \frac{\ell}{2^m}$. at most 1 step of size $\frac{k}{2^n} \to \frac{\ell'}{2^{m-1}}$ ...

$|t-s| < 2^{-m}$, we have $|X_t(\omega) - X_s(\omega)| \leq 2 \sum_{j=m+1}^{n} 2^{-\gamma j}$. **Pf:** Say $s, t \in \left[\frac{k}{2^m}, \frac{k+1}{2^m}\right)$.

Then $|X_t - X_{k/2^m}| \leq 2^{-\gamma n} + 2^{-\gamma(n-1)} \dots + 2^{-\gamma(m+1)} = \sum_{m+1}^{n} 2^{-\gamma j}$ & same for $|X_s - X_{k/2^m}|$. QED.

Claim ②: $\forall \omega \in \Omega^*$, $t \to X_t(\omega)$ is uniformly continuous for $t \in D$. (& loc Hölder cts with expont γ).

Pf: Let $h(\omega) = 2^{-n_0(\omega)}$. Say $|t-s| < h(\omega)$. Choose $m \ni 2^{-(m+1)} \leq |t-s| < 2^{-m}$. Note $m \geq n_0$

By claim ①, $|X_t - X_s| \leq 2 \sum_{m+1}^{\infty} 2^{-\gamma j} \leq 2\, 2^{-m\gamma} \leq 2^{\gamma+1}|t-s|^\gamma$ QED.

For BM: $E|B_t - B_s|^4 \leq C|t-s|^2$ $\Rightarrow B$ is locally C^α in time $\forall \alpha < 1/4$.

Now pick $t \in [0,1]$ & a sequence (d_n) of diadic rationals $\ni (d_n) \to t$.

Claim ② $\Rightarrow (X_{d_n}(\omega))$ is Cauchy & so converges to some $Y_t(\omega)$ [independent of the seq (d_n)].

$\forall \omega \notin \Omega^*$, set $Y_t = 0$. Note, further $\forall \omega \in \Omega^*$, $X_{d_n}(\omega) = Y_{d_n}(\omega)$, & since

$X_{d_n} \to X_t$ in probability $\Rightarrow Y_t = X_t$ a.s. Local Hölder continuity follows from Claim ② QED.

Last time: ① $\exists$ an adapted process $X = \{X_t, \mathcal{F}_t\} \ni$ ⓐ $X_0 = 0$ a.s., ⓑ $X_t - X_s \sim N(0, t-s)$ & is ind of $\mathcal{F}_s$

Continuity in time: Kolmogorov Chentsov: $E|X_t - X_s|^\alpha \le C(t-s)^{1+\beta} \Rightarrow X$ has a modification that is locally Holder cts with exp γ, $\forall \gamma < \beta/\alpha$.

For BM: $E(X_t - X_s)^2 = (t-s)^1$. Not enough. $E(X_t - X_s)^4 = c(t-s)^2 \Rightarrow$ Holder γ $\forall \gamma < \frac{1}{4}$.

In general $E|X_t - X_s|^N = c|t-s|^{N/2}$. Kolmogorov-Chentsov $\Rightarrow \exists$ a cts modification which is Holder cts with exp γ $\forall \gamma < (N/2 - 1)/N = \frac{1}{2} - 1/N$. How about Holder $1/2$?

Thm: (Law of Iterated Logarithm): W a BM. $\overline{\lim_{t\to 0^+}} \dfrac{W_t}{t^{1/2}\sqrt{2\ln|\ln t|}} = +1$ & $\underline{\lim_{t\to 0^+}} \dfrac{W_t}{t^{1/2}\sqrt{2\ln|\ln t|}} = -1$

Thm: Let $g(\delta) = \sqrt{2\delta \ln|\delta|}$. $P\left[\overline{\lim_{\delta\to 0^+}} \dfrac{1}{g(\delta)} \sup_{\substack{0\le s,t\le 1\\ |t-s|<\delta}} |W_t - W_s| = 1\right]$ a.s.

(g is an exact M.O.C. for B.M. on $[0,1]$).

Alternate construction of BM. Let $D_n = \{\frac{k}{2^n} \mid k \in \mathbb{N}\}$ & $D = \bigcup D_n$. Will construct $\{W_t \mid t \in D\} \ni$ ⓐ $W_0 = 0$ & ⓑ For $0 \le t_1 < \dots < t_n \in D$, $\{W_{t_1}, W_{t_2} - W_{t_1} \dots\} \sim N\left(0, \begin{pmatrix} t_1 & & \\ & t_2 - t_1 & \\ & & \ddots \end{pmatrix}\right)$

Then from K.C. $\tilde\forall \omega \in \Omega$, $f(t) = W_t(\omega) : D \to \mathbb{R}$ is Holder cts with exponent γ $\forall \gamma < 1/2$.

$\Rightarrow \forall t \in \mathbb{R}$, $\lim_{\substack{s\to t\\ s\in D}} W_s(\omega)$ exists a.s. Define $W_t(\omega) = \lim_{\substack{s\to t\\ s\in D}} W_s$. Immediately see that

ⓑ $\Rightarrow \forall\ 0 \le t_1 < \dots < t_n \in [0,\infty)$, $(W_{t_1}, W_{t_2} - W_{t_1}, \dots W_{t_n} - W_{t_{n-1}}) \sim N\left(0, \begin{pmatrix} t_1 & & \\ & t_2 - t_1 & \\ & & \ddots\ t_n - t_{n-1} \end{pmatrix}\right)$.

(b.c. a.s. conv $\Rightarrow$ conv in law). Let $\mathcal{F}_t = \mathcal{F}^W_t \overset{\text{IOU}}{\Rightarrow} W_t - W_s$ ind of $\mathcal{F}_s$.

Construction of W on D: Know $W_1 \sim N(0,1)$. & $W_1 = W_{1/2} + W_1 - W_{1/2}$ (each $N(0,1/2)$ & ind).

Let $\{N_{i,j}\}$ be a countable family of ind $N(0,1)$ R.V.

① For $t \in D_0 = \mathbb{N}$, define $W_t = \sum_1^t N_{i,0}$

② Let $W_{1/2} = \frac{1}{2}W_1 + \frac{1}{2}N_{1,1}$. Observe $W_1 - W_{1/2} = \frac{1}{2}W_1 - \frac{1}{2}N_{1,1}$

$\begin{pmatrix} W_{1/2} \\ W_1 - W_{1/2} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} W_1 \\ N_{1,1} \end{pmatrix} \Rightarrow (W_{1/2}, W_1 - W_{1/2})$ jointly normal.

$EW_{1/2} = E(W_1 - W_{1/2}) = 0$. & $E(W_{1/2}(W_1 - W_{1/2})) = E\left(\frac{1}{4}W_1^2 - \frac{1}{4}N_{11}^2\right) = 0$. $\Rightarrow W_1 - W_{1/2}$ ind of W_1.

③ Now inductively define $W_{\frac{2k+1}{2^{n+1}}} = W_{k/2^n} + \frac{1}{2}\left(W_{\frac{k+1}{2^n}} - W_{k/2^n}\right) + \frac{1}{2}\dfrac{N_{k,n+1}}{2^n}$

Above + induction $\Rightarrow \forall\ 0 \le t_1 < \dots < t_n \in D$, $(W_{t_1}, W_{t_2} - W_{t_1}, \dots W_{t_n} - W_{t_{n-1}}) \sim N\left(0, \begin{pmatrix} t_1 & & \\ & t_2 - t_1 & \\ & & \ddots\ t_n - t_{n-1} \end{pmatrix}\right)$ QED.

Prop: Say X is a process with independent increments (i.e. $(X_{t_0}, X_{t_1} - X_{t_0} \dots X_{t_n} - X_{t_{n-1}})$ are ind. Let $\mathcal{F}_t = \mathcal{F}^X_t$. Then $X_t - X_s$ is ind of $\mathcal{F}_s$.

Pf: Let $s < t$. $\mathcal{G} = \{A \in \mathcal{F}_s \mid X_t - X_s$ is ind of $A\}$. Let $\mathcal{C} = \{(X_{t_0}, X_{t_1} \dots X_{t_n}) \in A \mid t_i \le s$ & $A \in \mathcal{B}\}$.

$X_t - X_s$ ind of $(X_{t_0}, X_{t_1} - X_{t_0}, \dots X_{t_n} - X_{t_{n-1}}) \Rightarrow X_t - X_s$ ind of $(X_{t_0}, \dots X_{t_n}) \Rightarrow X_t - X_s$ ind of $\mathcal{C}$.

Claim: $\mathcal{C}$ is a π-system. $\mathcal{G}$ is a λ-system. ($\Rightarrow \mathcal{G} \supseteq \sigma(\mathcal{C}) = \mathcal{F}_s \Rightarrow$ QED (Prop)).

Pf: ① Say $A, B \in \mathcal{G}$ & $A \subseteq B$. NTS $B - A \in \mathcal{G}$. Pf: Let $C \in \sigma(X_t - X_s)$.

Then $P(C \cap (B - A)) = P((C\cap B) - (C \cap A)) = P(C)P(B) - P(C)P(A) = P(C)(P(B) - P(A)) \Rightarrow B - A \in \mathcal{G}$.

② $A_i \in \mathcal{G}$; $A_i \subseteq A_{i+1}$. NTS $\bigcup A_i \in \mathcal{G}$. Pf: $P(C \cap \bigcup A_i) = P(\bigcup(C \cap A_i)) = \lim P(C)P(A_i) \dots$ QED.

Prop: (Direct proof of Holder continuity). $\lim_{t\to s^+} \dfrac{|W_t - W_s|}{(t-s)^\alpha} = 0$ a.s.

Pf: $s = 0$ for simplicity. Pick $\varepsilon > 0$. Let $A_n = \left\{\sup_{2^{-(n+1)} \le t \le 2^{-n}} \dfrac{|W_t|}{t^\alpha} > \varepsilon\right\}$.

$P(A_n) \le P\left(\sup_{2^{-(n+1)} \le t \le 2^{-n}} |W_t| > \varepsilon 2^{\alpha n}\right) \le \dfrac{1}{\varepsilon 2^{\alpha n}} E W_{2^{-n}} \le \dfrac{c}{\varepsilon 2^{\alpha(n+1)}} \cdot 2^{-n/2} = \dfrac{c}{\varepsilon} 2^{-n(1/2 - \alpha)}$

$\Rightarrow \sum P(A_n) < \infty$. Borel Cantelli $\Rightarrow A_n$ don't occur i.o. $\Rightarrow \tilde\forall \omega \in \Omega$, $\exists n_0(\omega) \ni$

$\forall n > n_0(\omega)$, $\sup_{2^{-(n+1)} \le t \le 2^{-n}} \dfrac{|W_t(\omega)|}{t^\alpha} < \varepsilon$. $\Rightarrow \forall t \le 2^{-n_0(\omega)}$, $\dfrac{|W_t(\omega)|}{t^\alpha} < \varepsilon$. $\Rightarrow \lim_{t\to 0} \dfrac{|W_t(\omega)|}{t^\alpha} = 0$ a.s. QED.

Remk: Same proof shows that $\lim_{t\to+\infty} \left|\dfrac{W_t}{t^\alpha}\right| = 0$ a.s. $\forall \alpha > 1/2$. (You check).

Cor 1 Let W be a B.M. Let $B_t = t W_{1/t}$. Then $\{B_t, \mathcal{F}^B_t\}$ is a B.M.

Pf: $s < t$.
$(B_s, B_t - B_s) = \begin{pmatrix} sW_{1/s} \\ tW_{1/t} - sW_{1/s} \end{pmatrix} = \begin{pmatrix} s(W_{1/s} - W_{1/t}) + sW_{1/t} \\ -s(W_{1/s} - W_{1/t}) + (t-s)W_{1/t} \end{pmatrix} = \begin{pmatrix} s & s \\ -s & t-s \end{pmatrix} \begin{pmatrix} W_{1/t} \\ W_{1/s} - W_{1/t} \end{pmatrix}$

$\Rightarrow (B_s, B_t - B_s)$ Jointly normal. $E((B_t - B_s)B_s) = E\, sW_{1/s}(tW_{1/t} - sW_{1/s}) = \dfrac{st}{t} - \dfrac{s^2}{s} = 0$.

$E(B_t - B_s)^2 = t + s - 2\dfrac{st}{t} = t - s$. $\therefore$ N.C. $\Rightarrow B_t - B_s \sim N(0, t-s)$ & ind of B_s.

Induction $\Rightarrow (B_{t_0}, B_{t_1} - B_{t_0} \dots B_{t_n} - B_{t_{n-1}}) \sim N\left(0, \begin{pmatrix} t_0 & & \\ & t_1 - t_0 & \\ & & \ddots\ t_n - t_{n-1} \end{pmatrix}\right)$. Continuity in time by Cor 1.

QED