Stochastic Processes

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Motivation: ① Brownian Motion — Cts time random walk.
② Stochastic Integrals — Cts time martingale transform. (return on investment)
— Martingales are not B.V. Construction is non-trivial.
③ Itô formula. ④ Diffusions & connection to PDE. (Expected pay off satisfies...)

① **Notation:** Ω the sample space, with σ-alg $\mathcal{F}$ & measure P.

interpret as time

Def: A stochastic process is a collection $\{X_t \mid t \in [0,\infty)\}$, where $\forall t$,

$X_t : \Omega \to S$ is a random variable (S is some state space, usually $\mathbb{R}^d$).

For fixed $\omega \in \Omega$, the fn $t \to X_t(\omega)$ is called the trajectory / sample path of X

Def: ("Equality") Let X, Y be two stochastic processes.

① X, Y are said to be indistinguishable if $P\{X_t = Y_t \ \forall t\} = 1$

② X is said to be a modification of Y if $\forall t$, $P\{X_t = Y_t\} = 1$

③ X & Y have the same finite dim dist if $\forall n \in \mathbb{N}$, $A_1, \dots A_n \in \mathcal{B}(\mathbb{R}^d)$

& $t_1 \dots t_n \geq 0$, we have $P(X_{t_1} \in A_1, \dots X_{t_n} \in A_n) = P(Y_{t_1} \in A_1, \dots Y_{t_n} \in A_n)$

($\iff (X_{t_1} \dots X_{t_n})$ & $(Y_{t_1} \dots Y_{t_n})$ have the same law)

Remark: ① $\Rightarrow$ ② $\Rightarrow$ ③ Trivially. ③ $\not\Rightarrow$ ① or ② (eg. Let ξ be any symm R.V. with $P(\xi = 0) = 0$ & set $X_t = \xi$ & $Y_t = -\xi$) Also ② $\not\Rightarrow$ ①. Eg. Let τ be a +ve R.V. with continuous dist. Define $X_t = 0 \ \forall t$ & $Y_t = \chi_{\{t = \tau\}} = \begin{cases} 1 & t = \tau \\ 0 & t \neq \tau \end{cases}$

Then $\forall t \geq 0$, $P(X_t = Y_t) = P(Y_t = 0) = P(\tau \neq t) = 1. \Rightarrow X$ is a modification of Y.

But $P\{X_t = Y_t \ \forall t\} = P\{Y_t = 0 \ \forall t\} = P\{\tau \neq t \ \forall t\} = 0 \Rightarrow X, Y$ are not indistinguishable.

Remark: Say the trajectories of X & Y are right cts a.s. Then X, Y are indist $\iff$ X is a modification of Y. (Pf: $P\{X_t = Y_t \ \forall t \in \mathbb{Q}\} = 1 \Rightarrow P\{X_t = Y_t \ \forall t\} = 1$).

Def: We say $\{\mathcal{F}_t \mid t \in [0,\infty)\}$ is a filtration on $(\Omega, \mathcal{F})$ if $\forall t \geq 0$, $\mathcal{F}_t \subset \mathcal{F}$ is a σ-alg & $\forall s \leq t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$. Define $\mathcal{F}_\infty = \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t\right)$

Eg: Given a process X, define $\mathcal{F}_t^X = \sigma\left(\bigcup_{s \leq t} \sigma(X_s)\right)$ & $\mathcal{F}^X = \{\mathcal{F}_t^X \mid t \in [0,\infty)\}$.

(this the filtration generated by X. $\mathcal{F}_t^X \approx$ info gained up to time t).

Past — Infinitesimal peeks into the future.

Def: Let $\{\mathcal{F}_t\}$ be a filtration. Define $\mathcal{F}_{t^-} = \sigma\left(\bigcup_{s<t} \mathcal{F}_s\right)$ & $\mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s$

The filtration is right continuous if $\forall t$, $\mathcal{F}_{t^+} = \mathcal{F}_t$. (left cts if $\forall t$, $\mathcal{F}_{t^-} = \mathcal{F}_t$).

Def: ⓪ X is said to be measurable if $(t, \omega) \to X_t(\omega)$ is $\mathcal{B}([0,\infty)) \otimes \mathcal{F}$ meas

Def ① X is said to be prog-meas w.r.t. the filtration $\{\mathcal{F}_t\}$ if $\forall T$, $X|_{[0,T] \times \Omega}$ is $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$ meas ($\iff \{(t,\omega) \mid t \leq T \ \& \ X_t \in A\} \in \mathcal{F}_T \ \forall A \in \mathcal{B}(\mathbb{R}^d)$)

② X is said to be adapted to the filtration $\{\mathcal{F}_t\}$ if $\forall t$ X_t is $\mathcal{F}_t$ meas.

Remark: Prog meas $\Rightarrow$ adapted & meas. Hard thm: Adapted meas $\Rightarrow \exists$ a prog meas modification.

(Easy exercise: If sample paths are RC then adapted $\Rightarrow$ prog meas $\Rightarrow$ meas).

Stopping Times: Let $\tau : \Omega \to [0,\infty)$ be $\mathcal{F}$ meas. (called a random time)

Let $X_t = \begin{cases} 1 & t \geq \tau \\ 0 & t < \tau \end{cases}$. X is the process of "have you stopped at time t?" X adapted $\iff \tau$ a stopping time.

Def: τ a stopping time (wrt $\{\mathcal{F}_t\}$) if $\forall t$, $\{\tau \leq t\} \in \mathcal{F}_t$.

τ is an optional time (wrt the filtration $\{\mathcal{F}_t\}$) if $\forall t$, $\{\tau < t\} \in \mathcal{F}_t$.

Remark: Any stopping time is an optional time (Pf: $\{\tau < t\} = \bigcup_n \{\tau \leq t - 1/n\} \in \mathcal{F}_{t-1/n} \subseteq \mathcal{F}_t$)

Remark: $\mathcal{F}_t$ right continuous $\Rightarrow$ any optional time is a stopping time

Pf: $\{\tau \leq t\} = \bigcap \{\tau < t + 1/n\} \subseteq \bigcap_n \mathcal{F}_{t+1/n} = \mathcal{F}_{t^+} \overset{\text{R.C.}}{=} \mathcal{F}_t$ QED

Eg: (Exit time) $\{\mathcal{F}_t\}$ R.C., X cts & adapted, $D \subseteq \mathbb{R}^d$ open, $\tau = \inf\{t \geq 0 \mid X_t \notin D\}$ ← first exit time from D.

Then τ is a stopping time wrt $\{\mathcal{F}_t\}$.

Pf: ① Let $K_n \subseteq D$ be closed, $K_n \subseteq \mathring{K}_{n+1}$, & set $\tau_n = \tau_{K_n} = \inf\{s \mid X_s \notin K_n\}$.

② Claim: $\forall n$, τ_n is an optional time. (& hence stopping)

Pf: $\{\tau_n \geq t\} = \{X_s \in K_n \ \forall s \leq t\} \overset{\text{cont of traj \& K closed}}{=} \{X_q \in K_n \ \forall q \in \mathbb{Q}, q \leq t\} \in \mathcal{F}_t \Rightarrow \{\tau < t\} \in \mathcal{F}_t$.

③ τ_n is increasing & $\tau_n \to \tau$. (Pf: $\sigma = \sup_n \tau_n$. ① $\tau_n \leq \tau \ \forall n \Rightarrow \sigma \leq \tau$
② $X_{\tau_n} \notin \mathring{K}_m \ \forall m \leq n \Rightarrow X_\sigma \notin \bigcup_1^\infty \mathring{K}_m = \Omega \Rightarrow \sigma \geq \tau$) QED.

④ $\{\tau \leq t\} = \bigcap \{\tau_n \leq t\} \in \mathcal{F}_t$

③

Martingales.

Motivation: ① Quit at any time without changing the odds.
② Play up to time s. Signal 50 friends who play to time t. Don't change the odds.

Def: Let M_t be a process adapted to $\mathcal{F}_t$. ∋ $E|M_t| < \infty \ \forall t$

① $\{M_t, \mathcal{F}_t \mid t \in [0,\infty)\}$ is a sub martingale if $E(M_t \mid \mathcal{F}_s) \geq M_s$ "Life is a super martingale."
② " supermartingale " $\leq$
③ " martingale " $=$

"fair game" Best approx of M_t at time s is M_s.

Remark: Wrt $\{\mathcal{F}_t\}$ say M is a submart, M' is a martingale, M'' a super mart. are such that $M_T = M'_T = M''_T$. Then $M_t \leq M'_t \leq M''_t$ a.s. $\forall t \leq T$.

Pf: $M_t \leq E(M_T \mid \mathcal{F}_t) = E(M'_T \mid \mathcal{F}_t) = M'_t$

& $M''_t \geq E(M''_T \mid \mathcal{F}_t) = E(M'_T \mid \mathcal{F}_t) = M'_t$. QED.

Eg: Say X is $\mathcal{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$ measurable, & $E|X| < \infty$.
Then $M_t = E(X \mid \mathcal{F}_t)$ is a martingale wrt $\{\mathcal{F}_t\}$.

Eg If $\{A_t\}$ is an increasing adapted process with $E|A_t| \leq \infty \ \forall t$, $\{M_t, \mathcal{F}_t\}$ a mart. then $\{M_t + A_t, \mathcal{F}_t\}$ is a sub martingale & $\{M_t - A_t, \mathcal{F}_t\}$ is a super mart.

Prop: Let $\{X_t\}$ be a process adapted to $\{\mathcal{F}_t\}$ & $\varphi: \mathbb{R} \to \mathbb{R}$ ∋ $E|\varphi(X_t)| < \infty \ \forall t$.

① If X is a martingale & φ is convex then $\varphi(X)$ is a sub martingale.

② If X is a sub martingale & φ is convex & increasing, then $\varphi(X)$ is a sub-mart.

Pf: ① Pick $s < t$. Jensen $\Rightarrow E(\varphi(X_t) \mid \mathcal{F}_s) \geq \varphi(E(X_t \mid \mathcal{F}_s)) = \varphi(X_s)$
since X is a martingale. If X is a sub-martingale & φ is inc. get ineq above. QED.

Lemma: Let $\{X_t, \mathcal{F}_t\}_{t \geq 0}$ be a sub-martingale with R.C. trajectories.

① for $r < t$, $\lambda > 0$, $P(\sup_{r \leq s \leq t} X_s \geq \lambda) \leq \frac{1}{\lambda} E X_t^+$

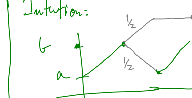
② (Doob's maximal inequality): If $p > 1$, & $X \geq 0$ then $E(\sup_{r \leq s \leq t} X_s^p) \leq (\frac{p}{p-1})^p E X_t^p$

Intuition: $\tau = \inf\{s \leq t \mid X_s^+ \geq \lambda\}$. $E X_t^+ \overset{OST}{\geq} E X_\tau^+ \geq \int_{\{\tau < t\}} X_\tau^+ = \lambda P(\sup_{r \leq s \leq t} X_s \geq \lambda)$

Pf: follows quickly from the corresponding discrete inequalities. Let $F = \{t_1 \cdots t_n\}$ where $r \leq t_1 < t_2 \cdots t_n \leq t$. Then $\{X_{t_i}, \mathcal{F}_{t_i} \mid i \leq n\}$ is a (discrete time) martingale & hence $P(\max_i X_{t_i} \geq \lambda) \leq \frac{1}{\lambda} E X_t^+$. Now write $\mathbb{Q} \cap [r,t]$ as a nested union of finite sets & get $P(\sup\{X_s \mid s \in (\mathbb{Q} \cap [r,t]) \cup \{r,t\}\} \geq \lambda) \leq \frac{1}{\lambda} E X_t^+$
Right continuity $\Rightarrow P(\sup\{X_s \mid s \in [r,t]\} \geq \lambda) \leq \frac{1}{\lambda} E X_t^+$ QED.

Def: Let X be a process, $F \subseteq [0,\infty)$ finite, $\alpha < \beta$. Let $\tau_1 = \min\{t \in F \mid X_t \leq \alpha\}$,
$\sigma_j = \min\{t \in F \mid t > \tau_j \ \& \ X_t > \beta\}$ & $\tau_{j+1} = \min\{t \in F \mid t > \sigma_j \ \& \ X_t \leq \alpha\}$.
Let $U_F(\alpha,\beta,X) = \max\{j \mid \sigma_j < \infty\}$ (# upcrossings).
If I is not finite, define $U_I(\alpha,\beta,X) = \sup_{\substack{F \subseteq I \\ F \text{ finite}}} U_F(\alpha,\beta,X)$.

Intuition:



$EU = 3/2. = \frac{E(X_t - a)^+}{b-a}$.

Thm: $X = \{X_t, \mathcal{F}_t\}$ an R.C. submartingale, $s < t$. Then
$$E\,U_{[s,t]}(\alpha,\beta,X) \leq \frac{E(X_t - \alpha)^+ - E(X_s - \alpha)^+}{\beta - \alpha} \leq \frac{E(X_t - \alpha)^+}{\beta - \alpha}$$

Remark: Once X is a R.C. submartingale, this will show that all left limits exist. (RCLL)

Def: The filtration is said to satisfy the usual conditions if it is right continuous and $\mathcal{F}_0$ contains all P-null sets in $\mathcal{F}$.

Thm: Let $X = \{X_t, \mathcal{F}_t\}$ be a submartingale & $\{\mathcal{F}_t\}$ satisfy the usual conditions.
Then the process X has a R.C. modification $\Leftrightarrow t \to E X_t$ is R.C.

Thm: $X = \{X_t, \mathcal{F}_t\}_{t < \infty}$ a RC sub-martingale. If $\sup_{t \geq 0} E X_t^+ < \infty$ then $\lim_{t \to \infty} X_t$ exists & $= X_\infty$ a.s. & $E X_\infty < \infty$ need not be a last elmt.

Pf: Let $C = \sup_{t \geq 0} E X_t^+$. Then $E\,U_{[0,\infty)}(\alpha,\beta,X) = \lim_{n \to \infty} E\,U_{[0,n]}(\alpha,\beta,X) \leq \lim_{n \to \infty} \frac{E X_n^+ + |\alpha|}{\beta - \alpha} \leq \frac{C + |\alpha|}{\beta - \alpha}$.
$\Rightarrow \forall \alpha, \beta$, $A_{\alpha,\beta} = \{U_{[0,\infty)}(\alpha,\beta,X) = \infty\}$ is a null set
$\Rightarrow \{\overline{\lim_{t \to \infty}} X_t > \underline{\lim_{t \to \infty}} X_t\} \subseteq \bigcup_{\alpha,\beta \in \mathbb{Q}} A_{\alpha,\beta}$ is a null set $\Rightarrow \lim_{t \to \infty} X_t$ exists a.s.
for integrability, $E X_\infty^+ \overset{\text{Fatou}}{\leq} \underline{\lim_{t \to \infty}} E X_t^+ \leq C$. Also $E X_t^- = E X_t^+ - E X_t \leq C - E X_0$.
$\therefore E X_\infty^- \leq \underline{\lim} E X_t^- \leq C - E X_0$ QED

Def: Let τ be a stopping time. Let $\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty \mid \forall t,\ A \cap \{\tau \le t\} \in \mathcal{F}_t\}$

Remark: X prog meas, then X_τ is $\mathcal{F}_\tau$ meas. Pf: $\{X_\tau \in U\} \cap \{\tau \le t\} = \{\omega \mid X|_{[0,t]\times\Omega}(\tau(\omega)\wedge t, \omega) \in U\} \cap \{\tau \le t\}$.
Since $X_\cdot(\omega)$ is Borel fn of time, $X|_{[0,t]\times\Omega}(\tau(\omega)\wedge t, \omega)$ is $\mathcal{F}_t$ meas. $\Rightarrow$ $\in \mathcal{F}_t$ QED.

Def: Let τ be optional. Define $\mathcal{F}_{\tau^+} = \{A \in \mathcal{F} \mid \forall t,\ A \cap \{\tau \le t\} \in \mathcal{F}_{t+}\}$

Thm: (Doob's Optional Sampling thm). Let $\{X_t, \mathcal{F}_t \mid 0 \le t \le \infty\}$ be a R.C. sub martingale with last element. Let $\sigma \le \tau$ be two optional times. Then $E(X_\tau \mid \mathcal{F}_{\sigma+}) \ge X_\sigma$.

Remark: Pf will show that if σ is a stopping time, $E(X_\tau \mid \mathcal{F}_\sigma) \ge X_\sigma$.

Cor: If X is a martingale with last element then $E(X_\tau) = EX_0$.

Lemma 1: Let $\{\mathcal{F}_n\}$ be a decreasing sequence of σ-alg, & $\{X_n, \mathcal{F}_n\}$ be a backward sub martingale (i.e. $E(X_n \mid \mathcal{F}_{n+1}) \ge X_{n+1}$). Then $\lim_{n\to\infty} EX_n > -\infty \Rightarrow \{X_n\}$ is U.I.

Remark: Main idea is that now Doob's inequality has $\lambda P(\sup_n X_n > \lambda) \le EX_1^+$ ← first term. Ind of n

Pf of Doob's OST: Let $\sigma_n = \begin{cases} \infty & \text{if } \sigma = \infty \\ k/2^n & \text{if } \sigma \in [\frac{k-1}{2^n}, \frac{k}{2^n}) \end{cases}$.

$\boxed{\{\sigma_n \le \tfrac{k}{2^n}\} = \{\sigma < k/2^n\} \Rightarrow \sigma_n \text{ is a stopping time}}$

Similarly $\tau_n = \begin{cases} \infty & \text{if } \tau = \infty \\ k/2^n & \text{if } \tau \in [\frac{k-1}{2^n}, k/2^n) \end{cases}$. Now σ_n, τ_n taken on only a discrete set of values. So discrete OST $\Rightarrow$ $E(X_{\tau_n} \mid \mathcal{F}_{\sigma_n}) \ge X_{\sigma_n}\ \forall n$. $\Rightarrow \forall A \in \mathcal{F}_{\sigma+} \subseteq \bigcap \mathcal{F}_{\sigma_n}$

we have. $\int_A X_{\sigma_n}\, dP \le \int_A X_{\tau_n}\, dP$. (NTS $\int_A X_\sigma \le \int_A X_\tau = \int_A E(X_\tau \mid \mathcal{F}_{\sigma+})$)

Now $\{X_{\sigma_n}, \mathcal{F}_{\sigma_n}\}$ is a backward martingale ($\because \sigma_n \downarrow \sigma$) & $EX_{\sigma_n} \ge EX_0$. So lemma 1 $\Rightarrow$ $\{X_{\sigma_n}\}$ is U.I. Since $X_{\sigma_n} \to X_\sigma$ (R.C. of X), U.I. $\Rightarrow$ X_σ is integrable & further $\lim_{n\to\infty} \int_A X_{\sigma_n}\, dP = \int_A X_\sigma\, dP$. III^{ly} $\lim \int_A X_{\tau_n} = \int_A X_\tau \Rightarrow \int_A X_\sigma \le \int_A X_\tau$ QED

Remark: Used the fact that X has a last element to use OST from discrete setting.

Prop: If we don't have a last element, but τ is bdd, then OST works.

Prop: Let $\{X_t, \mathcal{F}_t\}$ be a R.C. sub martingale, τ a stopping time. Then $X^\tau \overset{\text{def}}{=} \{X_{\tau\wedge t}, \mathcal{F}_t\}$ is a sub-M.
$X_{\tau\wedge t}$ is adapted.

Pf: Pick $s < t$, $A \in \mathcal{F}_s$. Then $\int_A X_{\tau\wedge t} = \int_{A\cap\{\tau \le s\}} X_{\tau\wedge t} + \int_{A\cap\{\tau > s\}} X_{\tau\wedge t}$.

① $\int_{A\cap\{\tau\le s\}} X_{\tau\wedge t} = \int_{A\cap\{\tau\le s\}} X_\tau = \int_{A\cap\{\tau\le s\}} X_{\tau\wedge s}$.

② Claim: $A \cap \{\tau > s\} \in \mathcal{F}_{\tau\wedge s}$. (Pf: Let $B_r = A \cap \{\tau > s\} \cap \{\tau\wedge s \le r\} = A \cap \{\tau > s\} \cap \{s \le r\}$.
ⓐ $r < s \Rightarrow B_r = \phi \in \mathcal{F}_r$. ⓑ $r \ge s$: $A, \{\tau > s\} \in \mathcal{F}_r \Rightarrow B_r \in \mathcal{F}_r$). $\overset{OST}{\Longrightarrow}$ $\int_{A\cap\{\tau>s\}} X_{\tau\wedge t} \ge \int_{A\cap\{\tau>s\}} X_{\tau\wedge s}$ QED.
$E(X_{\tau\wedge t} \mid \mathcal{F}_{\tau\wedge s}) \ge X_{\tau\wedge s}$

Def: $\{X_t, \mathcal{F}_t\}$ is called a local martingale if $\exists$ an increasing seq of stopping times $\{\tau_n\} \ni \tau_n \to \infty$ a.s & $\{X_{\tau_n\wedge t}, \mathcal{F}_t\}$ is a ~~continuous~~ martingale

Eg: Any martingale is a local martingale. Converse is false. ($\exists$ U.I. counter examples)
↑ any time discrete local mg is a mg. But cts time.

Notation: $\mathcal{M} = \{X \mid X \text{ is a R.C. martingale} \ \& \ X_0 = 0 \text{ a.s.}\}$. $\mathcal{M}_c = \{X \mid X \in \mathcal{M} \ \& \text{ has cts paths}\}$
$\mathcal{M}_{loc} = \{\ "\quad "\ \text{RC local martingale } "\ \}$ $\mathcal{M}_{c,loc} = \{X \mid X \in \mathcal{M}_{loc} \ \& \text{ has cts paths}\}$.

Prop: Let $M \in \mathcal{M}_{loc}$ & $E \sup_t |M_t| < \infty$. $\Rightarrow M \in \mathcal{M}$.

Pf: Know $\exists \tau_n \ni \{M_{\tau_n\wedge t}\} \in \mathcal{M}$, $\tau_n \le \tau_{n+1}$ & $\tau_n \to \infty$ a.s. Pick $s < t$.
Then $M_{\tau_n\wedge t} \to M_t$ a.s. Also $|M_{\tau_n\wedge t}| \le \sup_t |M_t| \in L^1(\Omega)$
so $M_{\tau_n\wedge t} \to M_t$ in L^1, $\Rightarrow$ $E(M_t \mid \mathcal{F}_s) = \lim E(M_{\tau_n\wedge t} \mid \mathcal{F}_s) = \lim M_{\tau_n\wedge s} = M_s$ QED.

Remark: If $M \in \mathcal{M}_{c,loc}$ then the localising sequence τ_n can be chosen to be $\tau_n = \tau(B(0,n))$ $\wedge n$

Def: Let $\mathcal{M}^2 = \{X \in \mathcal{M} \mid EX_t^2 < \infty\ \forall t\}$ & $\mathcal{M}^2_c = \{X \mid X \in \mathcal{M}^2 \ \& \text{ has continuous paths}\}$.

Def: $\forall t \ge 0$, $X \in \mathcal{M}^2$ define $\|X\|_t = (EX_t^2)^{1/2}$, and $\|X\| = \sum_n \frac{\|X\|_n \wedge 1}{2^n}$.
(Note $\|X\|_n \le \|X\|_{n+1}$ because X^2 is a sub martingale) Also, if $\|X - Y\| = 0$ then $\forall n \in \mathbb{N}$, $X_n = Y_n$. $\Rightarrow \forall t \in [n, n+1)$, $X_t = E(X_{n+1} \mid \mathcal{F}_t) = E(Y_{n+1} \mid \mathcal{F}_t) = Y_t$ a.s.
$\Rightarrow X, Y$ are indistinguishable by R.C. So $\|\ \|$ is a metric if we identify ind. processes.

Prop: $\mathcal{M}^2$ & $\mathcal{M}^2_c$ are complete metric spaces. ($\mathcal{M}^2_c$ is a closed subspace of $\mathcal{M}^2$).

Pf: Will only show M_c^2 is complete. ① $X^{(n)}$ a C.S. in M_c^2. Then $\forall t$,

$$E|X_t^{(n)} - X_t^{(m)}|^2 \le E|X_N^{(n)} - X_N^{(m)}|^2 \quad \forall N\in\mathbb{N},\ N>t. \Rightarrow \forall t,\ (X_t^{(n)}) \text{ is a C.S. in}$$

$L^2(\Omega, \mathcal{F}_t)$. $\therefore \exists X_t \ni \forall t$ $(X_t^{(n)}) \xrightarrow{L^2(\Omega,\mathcal{F}_t)} X_t$. $\Rightarrow (X_t^{(n)}) \xrightarrow{L^1} X_t$.

$\therefore E(X_t\mid\mathcal{F}_s) = \lim E(X_t^{(n)}\mid\mathcal{F}_s) = \lim X_s^{(n)} = X_s$. $\Rightarrow X$ is a martingale.

Also $E\sup_{t\le N}|X_t^{(m)} - X_t^{(n)}|^2 \le 4E|X_N^{(m)} - X_N^{(n)}|^2 \to 0 \Rightarrow \sup_{t\le N} |X_t^{(m_k)} - X_t^{(m_l)}|^2 \xrightarrow[k,l\to\infty]{a.s.} 0$

$\Rightarrow X$ has continuous sample paths a.s.

Quadratic Variation. Consider now processes defined on the interval $[0,T]$ for some $T>0$ fixed.

Let $\Delta = \{0=t_0, t_1, \dots, t_m = T\}$ with $t_i<t_{i+1}$ be a partition of $[0,T]$. Let $|\Delta| = \max_i t_{i+1}-t_i$.

If X is any (continuous) process, define $\langle X\rangle_t^\Delta = \sum_{i=0}^{m-1} (X_{t\wedge t_{i+1}} - X_{t\wedge t_i})^2$.

Note, $(X_{t\wedge t_{i+1}} - X_{t\wedge t_i})^2 = \begin{cases} 0 & t\le t_i \\ X_t - X_{t_i} & t\in[t_i,t_{i+1}) \\ X_{t_{i+1}} - X_{t_i} & t\ge t_{i+1}\end{cases}$

Def: Let Δ_n be a sequence of partitions with $|\Delta_n|\to 0$. If $\forall t$, $\lim_{n\to\infty}\langle X\rangle_t^{\Delta_n}$ exists (in probability) and is independent of the subsequence then the limit is called the quadratic variation of X, & denoted by $\langle X\rangle$.

Prop: Let X be a continuous process of B.V. Then $\langle X\rangle$ exists & is 0 a.s.

Pf: By assumption $c = \sup_\Delta \sum |X_{t_{i+1}\wedge t} - X_{t_i\wedge t}| < \infty$. Then $\langle X\rangle_t^\Delta = \sum (X_{t_{i+1}\wedge t} - X_{t_i\wedge t})^2$

$$\le \max_i |X_{t_{i+1}\wedge t} - X_{t_i\wedge t}| \sum |X_{t_{i+1}\wedge t} - X_{t_i\wedge t}| \le c\max_i|X_{t_{i+1}\wedge t} - X_{t_i\wedge t}| \xrightarrow{\text{unif cont}} 0 \text{ as } |\Delta|\to 0.$$ QED

Theorem: Let $M\in M_c$ be bounded. Then $\langle M\rangle$ exists, is a continuous, adapted, increasing process & $M^2 - \langle M\rangle \in M_c$. Further, if Δ_n is a seq of partitions with $|\Delta_n|\to0$, then $\langle M\rangle^{\Delta_n} \to \langle M\rangle$ in $L^2(\Omega, L^\infty[0,T])$.

Lemma 1 (Key) $\forall M\in M_c^2$, $EM_t^2 = E\langle M\rangle_t^\Delta$. Further $M^2 - \langle M\rangle^\Delta \in M_c$

Pf: $M_t^2 = M_t^2 - M_0^2 = \sum M_{t\wedge t_{k+1}}^2 - M_{t\wedge t_k}^2 = \sum (M_{t\wedge t_{k+1}} - M_{t\wedge t_k})^2 + 2M_{t\wedge t_{k+1}}M_{t\wedge t_k} - 2M_{t\wedge t_k}^2 =$

$$= \langle M\rangle_t^\Delta + 2\sum M_{t\wedge t_k}(M_{t\wedge t_{k+1}} - M_{t\wedge t_k}) = \langle M\rangle_t^\Delta + 2\sum M_{t_k}(M_{t\wedge t_{k+1}} - M_{t\wedge t_k})$$

Let $N_t^\Delta = M_t^2 - \langle M\rangle_t^\Delta = \sum M_{t_k}(M_{t\wedge t_{k+1}} - M_{t\wedge t_k})$. Claim: $N^\Delta\in M_c$ (Note Claim $\Rightarrow$ lemma).

Pf: ① For $s\le t_k$: $E(M_{t_k}(M_{t\wedge t_{k+1}} - M_{t\wedge t_k})\mid\mathcal{F}_s) = E(E(\quad\mid\mathcal{F}_{t_k})\mid\mathcal{F}_s)$

$$= E(M_{t_k}E(M_{t\wedge t_{k+1}} - M_{t\wedge t_k}\mid\mathcal{F}_{t_k})\mid\mathcal{F}_s) = 0 = M_{t_k}(M_{s\wedge t_{k+1}} - M_{s\wedge t_k})$$

② For $s>t_k$, $E(M_{t_k}(M_{t\wedge t_{k+1}} - M_{t\wedge t_k})\mid\mathcal{F}_s) = M_{t_k}E(\quad\mid\mathcal{F}_s) = M_{t_k}(M_{s\wedge t_{k+1}} - M_{s\wedge t_k})$ QED.

Cor: $\forall M\in M_c$, bdd, $E(N_t^\Delta)^2 \le \alpha^4$, where $\alpha = \sup_t|M_t|$. Hence $E(\langle M\rangle_t^\Delta)^2 \le 2(\alpha^4+\alpha^4)$

Pf: Let $j<k$, $t\ge t_k$. Then $E(M_{t_j}(M_{t_{j+1}\wedge t} - M_{t_j\wedge t})M_{t_k}(M_{t_{k+1}\wedge t} - M_{t_k\wedge t})) = EE(\quad\mid\mathcal{F}_{t_k}) = 0$.

For $t<t_k$, $M_{t_{k+1}\wedge t} - M_{t_k\wedge t} = 0$. $\Rightarrow E(N_t^\Delta)^2 = E\sum M_{t_k}^2(M_{t_{k+1}\wedge t} - M_{t_k\wedge t})^2 \le \alpha^2 E\langle M\rangle_t^\Delta \le \alpha^4$ QED

Lemma 2: $\forall M\in M_c$, bdd, Δ_n a seq of partitions with $|\Delta_n|\to0$, (N^{Δ_n}) is Cauchy in $M_c^2(0,T]$

Pf: Given a partition Δ, & a process M, define $f_t^\Delta = M_{t_k}$ if $t\in[t_k, t_{k+1})$.

Let $\Delta = \Delta_m\cup\Delta_n = \{0=u_0<u_1 \dots u_\ell = T\}$. Then $N_t^{\Delta_m} = \sum f_{u_k}^{\Delta_m}(M_{t\wedge u_{k+1}} - M_{t\wedge u_k})$

$$\Rightarrow N_t^{\Delta_m} - N_t^{\Delta_n} = \sum_k (f_{u_k}^{\Delta_m} - f_{u_k}^{\Delta_n})(M_{t\wedge u_{k+1}} - M_{t\wedge u_k})$$

$$\Rightarrow E(N_t^{\Delta_m} - N_t^{\Delta_n})^2 \overset{\text{cross terms cancel}}{=} E\sum (f^{\Delta_m}_{u_k} - f^{\Delta_n}_{u_k})^2 (M_{t\wedge u_{k+1}} - M_{t\wedge u_k})^2 \le E\sup_k (f^{\Delta_m}_{u_k} - f^{\Delta_n}_{u_k})^2 \langle M\rangle_t^\Delta$$

$$\le \left(E\sup_k(f^{\Delta_m}_{u_k} - f^{\Delta_n}_{u_k})^4\right)^{1/2}\left(E(\langle M\rangle_t^\Delta)^2\right)^{1/2} \le (2\alpha^4)^{1/2}\left(E\sup_k(f^{\Delta_m}_{u_k} - f^{\Delta_n}_{u_k})^4\right)^{1/2} \to 0$$

since $\sup_k (f^{\Delta_m}_{u_k} - f^{\Delta_n}_{u_k}) \xrightarrow{\text{continuity of } M} 0$ a.s. & is dominated by $2\alpha^4$ QED

Pf of thm: Since $M_t^2 = \langle M\rangle_t^\Delta + 2N_t^\Delta$, $(\langle M\rangle^{\Delta_n})$ converges $\Leftrightarrow$ (N^{Δ_n}) converges. By lemma 3, (N^{Δ_n}) is Cauchy in $M_c^2 \Rightarrow \exists N\in M_c^2 \ni (N^{\Delta_n}) \xrightarrow{M_c^2} N$. (Note that N is independent of the sequence of partitions Δ_n). Let $\langle M\rangle = M^2 - 2N$. Then

$E\sup_{s\le T}|\langle M\rangle_s^{\Delta_n} - \langle M\rangle_s|^2 = 4E\sup_{s\le T}|N_s^{\Delta_n} - N_s|^2 \le 16\|N^{\Delta_n} - N\|_T \to 0$. Note that $M^2 - 2N$ is continuous & adapted $\Rightarrow \langle M\rangle$ is cts + adapted. Also $\langle M\rangle^{\Delta_n}$ is increasing $\forall n$ $\Rightarrow \langle M\rangle$ is increasing. Finally $M^2 - \langle M\rangle = N \in M_c$. QED.

(4)

Recall: $\langle M\rangle^{\Delta}_t = \sum (M_{t_{i+1}\wedge t} - M_{t_i\wedge t})^2$. $M \in \mathcal{M}_c$, bdd $\Rightarrow \langle M\rangle^{\Delta} \to \langle M\rangle$ in $L^2(\Omega, L^\infty([0,T]))$. Also $M_t^2 = \underbrace{\langle M\rangle^{\Delta}}_{\to \langle M\rangle} + \underbrace{2N^{\Delta}}_{\to 2\int M\,dM}$.

Thm: Let $M \in \mathcal{M}_{c,loc}$. Then $\langle M\rangle$ exists & $M^2 - \langle M\rangle \in \mathcal{M}_{c,loc}$. Further if (Δ_n) is a seq of partitions with $|\Delta_n| \to 0$, then $\langle M\rangle^{\Delta_n} \to \langle M\rangle$ uniformly in time, in probability.

Pf: If τ is a stopping time, let $M^{\tau} = \{M_{t\wedge\tau}, \mathcal{F}_t\}$ be the stopped process.

Let $\{\tau_n\}$ be a localising sequence $\ni M^{\tau_n} \in \mathcal{M}_c$ is bdd $\forall n$. Then $\forall m \le n$, & partition Δ,

$$\langle M^{\tau_n}\rangle^{\Delta}_{t\wedge\tau_m} = \sum (M^{\tau_n}_{t_{i+1}\wedge\tau_m\wedge t} - M^{\tau_n}_{t_i\wedge\tau_m\wedge t})^2 = \langle M^{\tau_m}\rangle^{\Delta}_t \Rightarrow \langle M^{\tau_n}\rangle_{t\wedge\tau_m} = \langle M^{\tau_m}\rangle_t.$$

$\Rightarrow \exists$ a continuous, increasing, adapted process $\ni \langle M\rangle_t = \langle M^{\tau_n}\rangle_t$ $\forall t \le \tau_n$.

Now define a metric ρ by $\rho(X,Y) = E \sup_{t\le T} (X_t - Y_t)^2 \wedge 1$. Then

$$\rho(\langle M\rangle^{\Delta_m}, \langle M\rangle) \le E \sup_{t\le\tau_n} (\langle M^{\tau_n}\rangle^{\Delta_m} - \langle M^{\tau_n}\rangle)^2 \wedge 1 + P(\tau_n < T)$$

$$\le \rho(\langle M^{\tau_n}\rangle^{\Delta_m}, \langle M^{\tau_n}\rangle) + P(\tau_n < T) \text{ which can be made small.}$$

$\Rightarrow \rho(\langle M\rangle^{\Delta_m}, \langle M\rangle) \to 0 \Rightarrow \langle M\rangle^{\Delta_m} \to \langle M\rangle$ uniformly in time, in probability.

Also, $(M^2 - \langle M\rangle)^{\tau_n} = (M^{\tau_n})^2 - \langle M^{\tau_n}\rangle \in \mathcal{M}_c \Rightarrow M^2 - \langle M\rangle \in \mathcal{M}_{c,loc}$ QED.

Theorem: Let $M \in \mathcal{M}_{c,loc}$. Then $M \in \mathcal{M}_c^2 \iff \langle M\rangle$ is integrable. In this case $M^2 - \langle M\rangle \in \mathcal{M}_c$

Pf: ① Say $M \in \mathcal{M}_c^2$. $\Rightarrow M^2$ is a cts subm. $\Rightarrow E M_T^2 \ge E M^2_{\tau_n\wedge T} = E(M_T^{\tau_n})^2 = E\langle M^{\tau_n}\rangle_T$

But $\langle M^{\tau_n}\rangle_T = \langle M\rangle^{\tau_n}_T \xrightarrow{\text{monotone}} \langle M\rangle_T \Rightarrow E\langle M\rangle_T \le E M_T^2$. QED ①

② Say $E\langle M\rangle_T < \infty$. Pick (τ_n) a loc seq $\ni (M^{\tau_n})$ is bdd. Then $E \sup_{t\le T} M_t^2 \le \varliminf_{n\to\infty} E \sup_{t\le T} (M_t^{\tau_n})^2$

$$\le 4 \varliminf_{n\to\infty} E(M_T^{\tau_n})^2 = 4 \varliminf E\langle M\rangle^{\tau_n}_T \xrightarrow{\text{monotone}} 4E\langle M\rangle_T < \infty. \Rightarrow E \sup_{t\le T} |M_t| < \infty \text{ QED ②}$$

Finally know $M^2 - \langle M\rangle \in \mathcal{M}_{c,loc}$ & $E \sup_{t\le T} |M_t^2 - \langle M\rangle_t| \le 4E M_T^2 + E\langle M\rangle_T < \infty \Rightarrow M^2 - M \in \mathcal{M}_c$ QED.

Thm: Let $M \in \mathcal{M}_{c,loc}$ & A be continuous, increasing, adapted. Then $A = \langle M\rangle \iff M^2 - A \in \mathcal{M}_{c,loc}$

Rmk: Doob Meyer & k.s. def of q.v.

Pf: $\Leftarrow$: $M^2 - A \in \mathcal{M}_{c,loc}$ & $M^2 - \langle M\rangle \in \mathcal{M}_{c,loc} \Rightarrow A - \langle M\rangle \in \mathcal{M}_{c,loc}$. But $A - \langle M\rangle$ is B.V. & hence has 0 q.v. $\Rightarrow A - \langle M\rangle \in \mathcal{M}_c^2$ & $(A - \langle M\rangle)^2 - 0 \in \mathcal{M}_c \Rightarrow E(A_t - \langle M\rangle_t)^2 = 0$ $\forall t$ QED

Joint QV: Let $M, N \in \mathcal{M}_{c,loc}$. For any partition $\Delta = \{0 = t_0 < t_1 \cdots t_n = T\}$, define

$$\langle M,N\rangle^{\Delta}_t = \sum_0^{n-1} (M_{t\wedge t_{k+1}} - M_{t\wedge t_k})(N_{t\wedge t_{k+1}} - N_{t\wedge t_k}).$$

Define the joint q.v. $\langle M,N\rangle$ to be $\lim_{|\Delta_n|\to 0} \langle M,N\rangle^{\Delta_n}$, if the limit exists (in probability) & is ind of the seq of partitions.

Thm: If $M, N \in \mathcal{M}^c_{loc}$, then $\langle M,N\rangle$ exists & $MN - \langle M,N\rangle \in \mathcal{M}_{c,loc}$. Further if $|\Delta_n| \to 0$, then $\langle M,N\rangle^{\Delta_n} \to \langle M,N\rangle$ uniformly in time, in probability.

Pf: $\forall \Delta$, $\langle M,N\rangle^{\Delta} = \frac{1}{4}(\langle M+N\rangle^{\Delta} - \langle M-N\rangle^{\Delta})$, which immediately proves convergence & existence.

Also, $\langle M,N\rangle = \frac{1}{4}(\langle M+N\rangle - \langle M-N\rangle)$ & $MN = \frac{1}{4}((M+N)^2 - (M-N)^2)$

So $MN - \langle M,N\rangle = \frac{1}{4}[(M+N)^2 - \langle M+N\rangle] + \frac{1}{4}[(M-N)^2 - \langle M-N\rangle] \in \mathcal{M}_{c,loc}$. QED

Remark: As before, if $M, N \in \mathcal{M}_c^2$, then $\langle M,N\rangle$ is the unique, continuous, B.V. process $\ni MN - \langle M,N\rangle \in \mathcal{M}_c$

Def: We say $M, N \in \mathcal{M}_c^2$ are orthogonal if $\langle M,N\rangle = 0$

Note: Let $M, N \in \mathcal{M}_c^2$. Then $\langle M,N\rangle = 0 \iff \forall s < t$, $E((M_t - M_s)(N_t - N_s) \mid \mathcal{F}_s) = 0$.

conditionally uncorrelate inc.

Pf: Note $E((M_t - M_s)(N_t - N_s) \mid \mathcal{F}_s) = E(M_tN_t - M_sN_s \mid \mathcal{F}_s) = 0$ if $\langle M,N\rangle = 0$. ($\because MN - \langle M,N\rangle \in \mathcal{M}_c$)

Conversely $E(M_tN_t - M_sN_s \mid \mathcal{F}_s) = 0$ $\forall s < t \Rightarrow MN \in \mathcal{M}_c \Rightarrow \langle M,N\rangle = 0$. QED.

Note: $\langle\ \rangle$ satisfies ① $\langle M,M\rangle \ge 0$ $\forall M$, ② $\langle \alpha M + \beta M', N\rangle = \alpha\langle M,N\rangle + \beta\langle M',N\rangle$ & ③ $\langle M,N\rangle = \langle N,M\rangle$. This immediately $\Rightarrow$ "Cauchy Schwartz": $|\langle M,N\rangle|^2 \le \langle M\rangle\langle N\rangle$.

Translating in time also get $|\langle M,N\rangle_t - \langle M,N\rangle_s| \le (\langle M\rangle_t - \langle M\rangle_s)^{1/2} (\langle N\rangle_t - \langle N\rangle_s)^{1/2}$

Thm: (Kunita-Watanabe) $\exists\, \mathcal{N} \in \mathcal{F}$, $P(\mathcal{N}) = 0$ $\ni \forall \omega \notin \mathcal{N}$

$$\left| \int_0^T f_s g_s \, d\langle M,N\rangle_s \right| \le \left(\int_0^T f_s^2 \, d\langle M\rangle_s \right)^{1/2} \left(\int_0^T g_s^2 \, d\langle N\rangle_s \right)^{1/2} \quad \forall \text{ meas } f, g.$$

Pf: Say first $f_t = a_k$, $t \in [t_k, t_{k+1})$ & $g_t = b_k$, $t \in [t_k, t_{k+1})$

Then $\int_0^T f g \, d\langle M,N\rangle = \sum a_k b_k (\langle M,N\rangle_{t_{k+1}} - \langle M,N\rangle_{t_k}) \le \sum a_k b_k (\langle M\rangle_{t_{k+1}} - \langle M\rangle_{t_k})^{1/2} (\langle N\rangle_{t_{k+1}} - \langle N\rangle_{t_k})^{1/2}$

$$\le \left(\sum a_k^2 (\langle M\rangle_{t_{k+1}} - \langle M\rangle_{t_k})\right)^{1/2} \left(\sum b_k^2 (\langle N\rangle_{t_{k+1}} - \langle N\rangle_{t_k})\right)^{1/2} = \left(\int_0^T f^2 d\langle M\rangle\right)^{1/2} \left(\int_0^T g^2 d\langle N\rangle\right)^{1/2}.$$

Approximate by simple processes for the general case. QED.