880 Stochastic Calculus: Final.

Dec 11^{th}, 2013

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 3 hours. The exam has a total of 7 questions and 35 points.*
- *You may use any result from class or homework* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*
- *The questions are ROUGHLY in order of length / difficulty, and not in the order the material was covered. However, depending on your intuition, you might find a few of the later questions easier. Good luck!*

In this exam, Ω always denotes a probability space, with measure P. Brownian motion will usually be denoted by W or B, and the underlying filtration (if not explicitly mentioned) is denoted by $\mathcal{F} = \{\mathcal{F}_t\}_{t \geqslant 0}$, and is always assumed to satisfy the usual conditions.

$\boxed{5}$ 1. Let W be a 1-dimensional Brownian motion. True or false:

There exists a continuous martingale M and a continuous increasing process A so that $|W| = M + A$.

If yes, find M and A explicitly. If no, prove they don't exist.

$\boxed{5}$ 2. Let M be a continuous local martingale. Suppose further $E\langle M\rangle_t^2 < \infty$ for all $t \geqslant 0$. True or false:

The process M is necessarily a martingale.

Prove it, or find a counter example.

$\boxed{5}$ 3. Let M be a continuous local martingale. Suppose further $EM_t^2 < \infty$ for all $t \geqslant 0$. True or false:

The process M is necessarily a martingale.

Prove it, or find a counter example.

$\boxed{5}$ 4. Let X be a d-dimensional diffusion with drift b and diffusion matrix σ (i.e. $X_t = x + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s) dW_s$, with b, σ globally Lipshitz). Let $D \subseteq \mathbb{R}^d$ be a bounded smooth domain and define

$$u(x) = E^x \int_0^\tau e^{-\lambda s}\,ds, \quad \text{where } \lambda > 0 \text{ is deterministic, and } \tau = \inf\{t \geqslant 0 \bigm| X_t \notin D\}.$$

If $u \in C^2(D) \cap C(\bar{D})$, find a PDE and boundary conditions satisfied by u. [Hint: You know from your homework / class a PDE and boundary conditions for the function $v(x,t) = P^x(\tau \geqslant t)$. Feel free to assume that v is $C^2(D \times [0,\infty))$ and $v \in C_b(D \times (0,\infty))$ (and decays as $t \to \infty$).]

$\boxed{5}$ 5. True or false:

The limit $\displaystyle\lim_{t \to 1^-} (1-t) \int_0^t \frac{1}{1-s}\,dW_s$ exists almost surely.

Prove your answer. [This is a crucial step used to solve a question on your homework; please provide a complete proof and not simply say "done on homework".]

$\boxed{5}$ 6. Is the augmented filtration of Brownian motion is right continuous? Prove or disprove it. [This follows immediately from question on your homework; please provide a complete proof and not simply say "follows from homework".]

$\boxed{5}$ 7. Let $d > 1$, and W be a standard d-dimensional Brownian motion. Let $R = |W|$. Find a Brownian motion B, and functions b, σ such that

$$dR_t = b_t(R_t)\,dt + \sigma_t(R_t)\,dB_t.$$

Prove it. [This follows immediately from question on your homework; please provide a complete proof and not simply say "follows from homework". Since the proof of this is long, I will not penalize you for omitting the proofs of certain technical Lemmas *provided* your solution to this (and the other problems) convinces me that you will be able to prove these lemmas "cold" if asked.]