880 Stochastic Calculus: Final.

Mon Dec, 17^{th}, 2012

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 3 hours. The exam has a total of 7 questions and 70 points.*
- *You may use without proof any result that has been proved in class or on the homework, unless you are explicitly instructed otherwise. You must, however,* ***CLEARLY*** *state the result you are using.*
- *The first four questions are roughly in order of difficulty. The last three questions are homework questions, asked in order in which the material was covered in class.*

In this exam, Ω always denotes a probability space, with measure P. Brownian motion will usually be denoted by W or B, and the underlying filtration (if not explicitly mentioned) is denoted by $\mathcal{F} = \{\mathcal{F}_t\}_{t \geqslant 0}$, and is always assumed to satisfy the usual conditions.

1. *(10 points)* Let W be a standard m-dimensional Brownian motion, and $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Define the process X by $X_t = TW_t$. Find a necessary and sufficient condition on T so that X is a standard n-dimensional Brownian motion.

2. Let W be a standard 1-dimensional Brownian motion, and define the processes X, Y by
$$X_t = \int_0^t \chi_{\{W_s > 0\}} \, dW_s, \quad Y_t = \int_0^t \chi_{\{W_s \leqslant 0\}} \, dW_s.$$
   (a) *(5 points)* Compute the joint quadratic variation $\langle X, Y \rangle$.
   (b) *(5 points)* Are the processes X and Y independent? Prove / disprove.

3. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a bounded, globally Lipschitz function, and let X be the diffusion with drift b and diffusion matrix I (i.e. $X_t^x = x + \int_0^t b(X_s^x) \, ds + W_t$, where W is a standard d-dimensional Brownian motion).
   (a) *(5 points)* Let $x \in \mathbb{R}^d$, $t > 0$ and choose $A \in \mathcal{B}(\mathbb{R}^d)$ with non zero Lebesgue measure. Do we necessarily have $P(X_t^x \in A) > 0$? Prove it, or find a counter example.
   (b) *(3 points)* Let $f : \mathbb{R}^d \to \mathbb{R}$, be Borel measurable and bounded and define $u(x,t) = Ef(X_t^x)$. State and prove a necessary and sufficient condition on f which guarantees
$$u(x,t) < \operatorname*{ess\,sup}_{y \in \mathbb{R}^d} f(y), \qquad \text{for all } x \in \mathbb{R}^d, t > 0.$$
   Recall $\operatorname{ess\,sup} f = \inf\{\lambda \in \mathbb{R} \mid |\{f > \lambda\}| = 0\}$, where $|\{f > \lambda\}|$ denotes the Lebesgue measure of the set $\{x \mid f(x) > \lambda\}$.
   (c) *(2 points)* If additionally $u \in C_0^{2,1}$, state (without proof) a PDE (and initial data) satisfied by the function u. [For your own interest: What is the PDE analogue of the result you proved in the previous subpart?]

4. *(10 points)* Let W be a standard 1-dimensional Brownian motion with $W_0 = 0$. Let $\sigma = \inf\{t \geqslant 0 \mid W_t = 1\}$ be the first time W hits 1. Let $\tau = \inf\{t > \sigma \mid W_t = -1\}$ be the first time *after* σ that W hits -1. Compute $P(\tau < t)$. [Note: You may leave your answer as an unsimplified (deterministic) integral, provide the integrand is an explicitly computable function.]

5. *(10 points)* Let $M = \{M_t, \mathcal{F}_t \mid t \in [0,T)\}$ be a square integrable, continuous martingale with $M_0 = 0$ almost surely. Let $\langle M \rangle_{T^-}$ denote $\lim_{t \to T^-} \langle M \rangle_t$ (which exists since $\langle M \rangle$ is an increasing function of time). Compute
$$P\left(\{\langle M \rangle_{T^-} < \infty\} \Delta \left\{ \lim_{t \to T^-} M_t \text{ exists, and is finite} \right\} \right),$$
   and prove your answer. Here $A \Delta B = A \cup B - A \cap B$ denotes the symmetric difference between A and B. [This was a question on HW#2. Please provide a complete solution here, and don't simply say "done on homework".]

6. *(10 points)* Is the process X, defined by $X_t \stackrel{\text{def}}{=} |W_t|$ a Markov process? Prove / disprove. Additionally compute (with proof) the transition density of the process X. [When showing X is a Markov process, you may choose the filtration to be either the (augmented) Brownian filtration, or the (augmented) filtration generated by the process X, whichever you find more convenient. Also note that this was a question on HW#3. Please provide a complete solution here, and don't simply say "done on homework".]

7. *(10 points)* Let b, σ satisfy the usual uniform Lipschitz and linear growth conditions. That is, suppose there exists a constant c such that

$$|b_t(x) - b_t(y)| + |\sigma_t(x) - \sigma_t(y)| \leqslant c|x - y| \quad \text{and} \quad |b_t(x)| + |\sigma_t(x)| \leqslant c(1 + |x|),$$

for all $x, y \in \mathbb{R}^d$. Suppose μ is a probability measure with finite variance. Do any two (weak or strong) solutions to the SDE

$$dX_t = b_t(X_t)\, dt + \sigma_t(X_t)\, dW_t$$

with initial distribution μ have the same law? Prove or find a counter example. [This was a question on HW#5. Please provide a complete solution here, and don't simply say "done on homework".]