Math 720: Homework.

Do, but don't turn in optional problems. *There is a firm 'no late homework' policy.*

Assignment 1: Assigned Wed 08/28. Due Wed 09/04

Keep in mind there is a firm "no late homework" policy. Starred problems are optional; but I'd recommend looking at them. They often involve results I will use later in class.

1. Let μ be a positive measure on (X, Σ).
   (a) If $A_i \in \Sigma$ are such that $A_i \subseteq A_{i+1}$, show that $\mu(\bigcup_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$.
   (b) If $A_i \in \Sigma$ are such that $A_i \supseteq A_{i+1}$, show that $\mu(\bigcap_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$, provided $\mu(A_1) < \infty$. Show by example this is false true if $\mu(A_1) = \infty$.
2. Prove any open subset of $\mathbb{R}^d$ is a countable union of cells.
3. For each of the following sets, compute the Lebesgue outer measure.
   (a) Any countable set. (b) The Cantor set. (c) $\{x \in [0,1] \mid x \notin \mathbb{Q}\}$.
4. (a) If $V \subseteq \mathbb{R}^d$ is a subspace with $\dim(V) < d$, then show that $\lambda^*(V) = 0$.
   (b) If $P \subseteq \mathbb{R}^2$ is a polygon show that $\text{area}(P) = \lambda^*(P)$.
5. Does there exist a σ-algebra whose cardinality is countably infinite? Disprove, or find an example.

Optional problems, and details in class I left for you to check.

- ∗ Define $\mu(A)$ to be the number of elements in A. Show that μ is a measure on $(X, \mathcal{P}(X))$. (This is called the counting measure.)
- ∗ Let $x_0 \in X$ be fixed. Define $\delta_{x_0}(A) = 1$ if $x_0 \in A$ and 0 otherwise. Show that δ_{x_0} is a measure on $(X, \mathcal{P}(X))$. (This is called the delta measure at x_0.)
- ∗ Show that $\lambda^*(a + E) = \lambda^*(E)$ for all $a \in \mathbb{R}^d$, $E \subseteq \mathbb{R}^d$.
- ∗ Show that $\lambda^*(I) = \ell(I)$ for all cells. (I only proved it for closed cells in class.)
- ∗ Show that $\mathcal{B}(\mathbb{R})$ has the same cardinality as $\mathbb{R}$.
- ∗ *(Challenge)* Suppose $f_n : [0,1] \to [0,1]$ are all Riemann integrable, $0 \leqslant f_n \leqslant 1$ and $(f_n) \to 0$ pointwise. Show that $\lim\limits_{n\to\infty} \int_0^1 f_n = 0$, using only standard tools from Riemann integration.

Assignment 2: Assigned Wed 09/04. Due Wed 09/11

1. (a) Say μ is a *translation invariant* measure on $(\mathbb{R}^d, \mathcal{L})$ (i.e. $\mu(x + A) = \mu(A)$ for all $A \in \mathcal{L}$, $x \in \mathbb{R}^d$) which is finite on bounded sets. Show that $\exists c \geqslant 0$ such that $\mu(A) = c\lambda(A)$.
   (b) Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a linear transformation, and $A \in \mathcal{L}$. Show that $T(A) \in \mathcal{L}$ and $\lambda(T(A)) = |\det(T)|\lambda(A)$. [Hint: Express T in terms of elementary transformations.]
2. (a) Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset \in \mathcal{E}$, $X \in \mathcal{E}$ and $\rho(\emptyset) = 0$. For any $A \subseteq X$ define
   $$\mu^*(A) = \inf\Big\{\sum_1^\infty \rho(E_i) \Bigm| E_i \in \mathcal{E}, \text{ and } A \subseteq \bigcup_1^\infty E_j\Big\}.$$
   Show that μ^* is an outer measure.
   (b) Let (X, d) be any metric space, $\delta > 0$, $\alpha \geqslant 0$ and define
   $$\mathcal{E}_\delta = \{A \subseteq X \mid \text{diam}(A) < \delta\} \quad\text{and}\quad \rho_\alpha(A) = \frac{\pi^{\alpha/2}}{\Gamma(1 + \frac{\alpha}{2})}\Big(\frac{\text{diam}(A)}{2}\Big)^\alpha.$$
   Let $H^*_{\alpha,\delta}$ be the outer measure obtained with $\rho = \rho_\alpha$ and the collection of sets $\mathcal{E}_\delta$. Define $H^*_\alpha = \lim_{\delta\to0} H^*_{\alpha,\delta}$. Show H^*_α is an outer measure and restricts to a measure H_α on a σ-algebra that contains all Borel sets. The measure H_α is called the *Hausdorff measure of dimension* α.
   (c) If $X = \mathbb{R}^d$, and $\alpha = d$ show that H_d is a non-zero, finite constant multiple of the Lebesgue measure. [In fact $H_d = \lambda$ because of our choice of normalization constant, but the proof is much harder.]
   (d) Let $S \in \mathcal{B}(X)$. Show that there exists (a unique) $d \in [0, \infty]$ such that $H_\alpha(S) = \infty$ for all $\alpha \in (0, d)$, and $H_\alpha(S) = 0$ for all $\alpha \in (d, \infty)$. This number is called the *Hausdorff dimension* of the set S.
   (e) Compute the Hausdorff dimension of the Cantor set.
3. Using notation from the previous question, let $\mathcal{S}_\delta = \{B(x, r) \mid x \in X, r \in (0, \delta)\}$. Using the collection of sets $\mathcal{S}_\delta$ and the function $\rho = \rho_\alpha$, we obtain an outer measure $S^*_{\alpha,\delta}$. As before one can show that $S^*_\alpha = \lim_{\delta\to0} S^*_{\alpha,\delta}$ is an outer measure, and gives a Borel measure S_α.
   (a) Show by example $S_\alpha \neq H_\alpha$ in general.
   (b) If $X = \mathbb{R}^d$ with the standard metric show that $S_d = \lambda$. [You may assume $\rho_d(B_r) = \lambda(B_r)$.]

Details in class I left for you to check. (Do it, but don't turn it in.)

- ∗ Using notation from the proof of Caratheodory, show that $\mu^*(A \cap (\cup_1^\infty E_i)) = \sum_1^\infty \mu^*(A \cap E_i)$. [We only proved it for $A = X$ in class.]

Assignment 3: Assigned Wed 09/11. Due Wed 09/18

1. Let μ, ν be two measures on (X, Σ). Suppose $\mathcal{C} \subseteq \Sigma$ is a π-system such that $\mu = \nu$ on $\mathcal{C}$.
   (a) Suppose $\exists C_i \in \mathcal{C}$ such that $\bigcup_1^\infty C_i = X$ and $\mu(C_i) = \nu(C_i) < \infty$. Show that $\mu = \nu$ on $\sigma(\mathcal{C})$.
   (b) If we drop the finiteness condition $\mu(C_i) < \infty$ is the previous subpart still true? Prove or find a counter example.
2. (a) Let X be a metric space and μ a Borel measure on X. Suppose there exists a sequence of sets $B_n \subseteq X$ such that $\bar{B}_n \subseteq \mathring{B}_{n+1}$, $\bar{B}_n$ is compact, $X = \cup_1^\infty B_n$ and $\mu(B_n) < \infty$. Show that μ is regular.
   (b) Show directly that for all $A \in \mathcal{L}$, $\lambda(A) = \sup\{\lambda(K)\}$ where $K \subseteq A$ is compact, and $\lambda(A) = \inf\{\lambda(U)\}$ where $U \supseteq A$ is open. [Note: The previous subpart will *only* show this for all $A \in \mathcal{B}(\mathbb{R}^d)$.]
3. (a) Find $E \in \mathcal{B}(\mathbb{R})$ so that for all $a < b$, we have $0 < \lambda(E \cap (a,b)) < b - a$.
   (b) Let $\kappa \in (0, 1/2)$. Does there exist $E \in \mathcal{B}(\mathbb{R})$ such that for all $a < b \in \mathbb{R}$, we have $\kappa(b-a) \leqslant \lambda(I \cap (a,b)) \leqslant (1-\kappa)(b-a)$? Prove it.
4. Let $A \in \mathcal{L}(\mathbb{R}^d)$. Prove every subset of A is Lebesgue measurable $\iff \lambda(A) = 0$.
5. (a) Prove $\mathcal{B}(\mathbb{R}^{m+n}) = \sigma(\{A \times B \mid A \in \mathcal{B}(\mathbb{R}^m) \,\&\, B \in \mathcal{B}(\mathbb{R}^n)\})$.
   (b) Prove $\mathcal{L}(\mathbb{R}^{m+n}) \supsetneq \sigma(\{A \times B \mid A \in \mathcal{L}(\mathbb{R}^m) \,\&\, B \in \mathcal{L}(\mathbb{R}^n)\})$.
   (c) Show $\mathcal{L}(\mathbb{R}^2) \supsetneq \mathcal{B}(\mathbb{R}^2)$.

Optional problems, and details in class I left for you to check.

* Let μ be a finite Borel measure on a compact metric space. Let
$$\mathcal{C} = \{A \mid \sup_{\substack{K \subseteq A \\ K \text{ compact}}} \mu(K) = \mu(A) = \inf_{\substack{U \supseteq A \\ U \text{ open}}} \mu(U)\}.$$
  We saw in class that $\mathcal{C}$ is closed under countable increasing unions. Show $\mathcal{C}$ is closed under relative compliments.
* Is any σ-finite Borel measure on $\mathbb{R}^d$ regular?
* Show that there exists $A \subseteq \mathbb{R}$ such that if $B \subseteq A$ and $B \in \mathcal{L}$ then $\lambda(B) = 0$, *and further*, if $B \subseteq A^c$ and $B \in \mathcal{L}$ then $\lambda(B) = 0$.

We say $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra if $\emptyset \in \mathcal{A}$, and $\mathcal{A}$ is closed under complements and *finite* unions. We say $\mu_0 : \mathcal{A} \to [0, \infty]$ is a (positive) *pre-measure* on $\mathcal{A}$ if $\mu_0(\emptyset) = 0$, and for any countable disjoint sequence of sets sequence $A_i \in \mathcal{A}$ *such that* $\bigcup_1^\infty A_i \in \mathcal{A}$, we have $\mu_0(\bigcup_1^\infty A_i) = \sum_1^\infty \mu_0(A_i)$.

Namely, a pre-measure is a finitely additive measure on an algebra $\mathcal{A}$, which is also countably additive for disjoint unions *that belong to the algebra*.

* *(Caratheodory extension)* If $\mathcal{A}$ is an algebra, and μ_0 is a pre-measure on $\mathcal{A}$, show that there exists a measure μ defined on $\sigma(\mathcal{A})$ that extends μ_0.

Assignment 4: Assigned Wed 09/18. Due Wed 09/25

1. Let $C \subseteq \mathbb{R}^d$ be convex. Must C be Lebesgue measurable? Must C be Borel measurable? Prove or find counter examples. [The cases $d = 1$ and $d > 1$ are different.]
2. Let (X, Σ, μ) be a measure space. For $A \in P(X)$ define $\mu^*(A) = \inf\{\mu(E) \mid E \supseteq A \,\&\, E \in \Sigma\}$, and $\mu_*(A) = \sup\{\mu(E) \mid E \subseteq A \,\&\, E \in \Sigma\}$.
   (a) Show that μ^* is an outer measure.
   (b) Let $A_1, A_2, \dots \in \mathcal{P}(X)$ be disjoint. Show that $\mu_*(\bigcup_1^\infty A_i) \geqslant \sum_1^\infty \mu_*(A_i)$. [The set function μ_* is called an *inner measure*.]
   (c) Show that for all $A \subseteq X$, $\mu^*(A) + \mu_*(A^c) = \mu(X)$.
   (d) Let $A \subseteq \mathcal{P}(X)$ with $\mu^*(A) < \infty$. Show that $A \in \Sigma_\mu \iff \mu_*(A) = \mu^*(A)$.
3. Let $f : X \to \mathbb{R}$ be measurable, and $g : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. True or false: $g \circ f : X \to \mathbb{R}$ is measurable? Prove or find a counter example.
4. Let (X, Σ) be a measure space, and $f, g : X \to [-\infty, \infty]$ be measurable. Suppose whenever $g = 0$, $f \neq 0$, and whenever $f = \pm\infty$, $g \in (-\infty, \infty)$. Show that $\frac{f}{g} : X \to [-\infty, \infty]$ is measurable. [Note that by the given data you will never get a 'meaningless' quotient of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. The remainder of the quotients (e.g. $\frac{1}{\infty}$) can be defined in the natural manner.]
5. Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions such that $(f_n) \to f$ almost everywhere (a.e.). Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function.
   (a) If for a.e. $x \in X$, g is continuous at $f(x)$, then show $(g \circ f_n) \to g \circ f$ a.e.
   (b) Is the previous part true without the continuity assumption on g?

Optional problems, and details in class I left for you to check.

* *(An alternate approach to λ-systems.)* Let $\mathcal{M} \subseteq P(X)$. We say $\mathcal{M}$ is a *Monotone Class*, if whenever $A_i, B_i \in \mathcal{M}$ with $A_i \subseteq A_{i+1}$ and $B_i \supseteq B_{i+1}$ then $\bigcup_1^\infty A_i \in \mathcal{M}$ and $\bigcap_1^\infty B_i \in \mathcal{M}$. If $\mathcal{A} \subseteq P(X)$ is an algebra, then show that the *smallest* monotone class containing $\mathcal{A}$ is exactly $\sigma(\mathcal{A})$. [You should also address existence of a smallest monotone class containing $\mathcal{A}$.]
* Prove that the completion Σ_μ we defined in class is the smallest μ-complete σ-algebra that contains Σ.
* Show that $f : X \to [-\infty, \infty]$ is measurable if and only if any of the following conditions hold
  (a) $\{f < a\} \in \Sigma$ for all $a \in \mathbb{R}$.
  (b) $\{f > a\} \in \Sigma$ for all $a \in \mathbb{R}$.
  (c) $\{f \leqslant a\} \in \Sigma$ for all $a \in \mathbb{R}$.
  (d) $\{f \geqslant a\} \in \Sigma$ for all $a \in \mathbb{R}$.
* Let (f_n) is a sequence of real valued measurable functions. Define $f(x) = \lim f_n(x)$ if the limit exists, and $f(x) = \infty$ otherwise. Show that f is measurable.

Assignment 5: Assigned Wed 09/25. Due Wed 10/02

1. Let (X, Σ, μ) be a measure space, and $(X, \Sigma_\mu, \bar{\mu})$ it's completion. Show that $g : X \to [-\infty, \infty]$ is Σ_μ-measurable if and only if there exists two Σ-measurable functions $f, h : X \to [-\infty, \infty]$ such that $f = h$ μ-almost everywhere, and $f \leqslant g \leqslant h$ everywhere.
2. Let μ be a regular (but not necessarily finite) Borel measure on a metric space X.
   (a) True or false: For any $f : X \to \mathbb{R}$ measurable and $\varepsilon > 0$ there exists $g : X \to \mathbb{R}$ continuous such that $\mu\{f \neq g\} < \varepsilon$? Prove it or find a counter example.
   (b) Do the previous subpart when $X = \mathbb{R}^d$.
3. Let for $n \in \mathbb{N}$ define $A_n = \bigcup_{k \in \mathbb{Z}} [\frac{2k}{2^n}, \frac{2k+1}{2^n})$. If $E \in \mathcal{B}(\mathbb{R})$ does $\lim_{n \to \infty} \lambda(A_n \cap E)$ exist? Prove it.
4. If $f \geqslant 0$ is measurable show that $\int_X f \, d\mu = 0 \iff f = 0$ almost everywhere.
5. (a) Suppose $I \subseteq \mathbb{R}^d$ is a cell, and $f : I \to \mathbb{R}$ is Riemann integrable. Show that f is measurable, Lebesgue integrable and that the Lebesgue integral of f equals the Riemann integral.
   (b) Is the previous subpart true if we only assume that an improper (Riemann) integral of f exists? Prove or find a counter example.

Optional problems, and details in class I left for you to check.

* Let $f : [0, 1] \to [0, 1]$ be the Cantor function, and $g(x) = \inf\{f = x\}$. Show that f is (Hölder) continuous, and the range of g is the Cantor set. What is the largest exponent α for which f is Hölder-α continuous?
* Let μ be the counting measure on $\mathbb{N}$, and $f : \mathbb{N} \to \mathbb{R}$ a function.
  (a) If $\sum_1^\infty |f(n)| < \infty$, then show that $\sum_{n=1}^\infty f(n) = \int_{\mathbb{N}} f \, d\mu$.
  (b) If the series $\sum_{n=1}^\infty f(n)$ is conditionally convergent, show that $\int_{\mathbb{N}} f \, d\mu$ is not defined.
* Let X be a metric space $C \subseteq X$ be closed and $f : C \to \mathbb{R}$ be continuous.
  (a) If $0 \leqslant f \leqslant 1$, then show that there exists $F : X \to \mathbb{R}$ continuous such that $F(c) = f(c)$ for all $c \in C$. [Hint: Let $F(x) = f(x)$ for all $x \in C$, and $F(x) = \inf\{f(c) + \frac{d(x,c)}{d(x,C)} - 1 \mid c \in C\}$ for $x \notin C$.]
  (b) *(Tietze extension theorem in metric spaces)* Do the previous subpart without assuming $0 \leqslant f \leqslant 1$. [Hint: Put $g = \tan^{-1}(f)$, construct G by the previous subpart and set $F = \tan(G)$.]
* Finish the proof of Lusin's theorem. (I only proved it for bounded positive functions in class.)
* Find a Borel measurable function $f : [0, 1] \to \mathbb{R}$ which is not continuous almost everywhere.
* Let $0 \leqslant s \leqslant t$ be two simple functions. Show $\int_X s \leqslant \int_X t$.
* Show directly $\int_X \alpha f = \alpha \int_X f$ for any $\alpha \in \mathbb{R}$ and integrable function f.

Assignment 6: Assigned Wed 10/02. Due Never

In light of your ***MIDTERM*** *this homework is optional.*

1. (a) If f is a bounded measurable function and $\mu(X) < \infty$, then show $\int_X f d\mu = \inf\{\int_X t \, d\mu \mid t \geqslant f \text{ is simple}\}$.
   (b) If f, g are bounded measurable functions and $\mu(X) < \infty$ show directly that $\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$.
2. Let $f : [0, \infty) \to \mathbb{R}$ be a measurable function. We define the *Laplace Transform* of f to be the function $F(s) = \int_0^\infty \exp(-st) f(t) \, dt$ wherever defined.
   (a) If $\int_0^\infty |f(t)| \, dt < \infty$, show that $F : [0, \infty) \to \mathbb{R}$ is continuous.
   (b) If $\int_0^\infty t|f(t)| \, dt < \infty$, show that $F : [0, \infty) \to \mathbb{R}$ is differentiable.
   (c) If f is continuous and bounded, compute $\lim_{s \to \infty} sF(s)$.
3. For $p \in \mathbb{R}$ define define $F(y) = \displaystyle\int_0^\infty \frac{\sin(xy)}{1 + x^p} \, dx$.
   (a) For what $p \in \mathbb{R}$ is F defined? When defined, is F continuous? Prove it.
   (b) Show that F is differentiable for $p > 2$, and not differentiable when $p = 2$.
4. *(Push forward measures)* Let μ be a measure on (X, Σ), and $f : X \to Y$ be any function. Define $\tau \subseteq \mathcal{P}(Y)$ by $\tau = \{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$. For $A \in \tau$ define $\nu(A) = \mu(f^{-1}(A))$.
   (a) Show that τ is a σ-algebra, and ν is a measure on (Y, τ). [The measure ν is called the push-forward of μ under f, and often denoted by $\mu_{f^{-1}}$.]
   (b) If $g \in L^1(Y, \nu)$, then show that $g \circ f \in L^1(X, \mu)$ and $\int_X g \circ f \, d\mu = \int_Y g \, d\nu$.
5. *(Pull back measures)* Say ν is a measure on (Y, τ) and $f : X \to Y$ is surjective.
   (a) Show that $\Sigma = \{A \subseteq X \mid f(A) \in \tau\}$ need not be a σ-algebra. If Σ is a σ-algebra, show that $\mu(A) = \nu(f(A))$ need not be a measure on (X, Σ).
   (b) Define instead $\Sigma = \{A \subseteq X \mid f^{-1}(f(A)) = A, \& f(A) \in \tau\}$, and $\mu(A) = \nu(f(A))$. Show that Σ is a σ-algebra and μ is a measure.
   (c) If $g \in L^1(Y, \nu)$, then show that $g \circ f \in L^1(X, \mu)$ and $\int_X g \circ f \, d\mu = \int_Y g \, d\nu$.
6. *(Linear change of variable)* Let $f : \mathbb{R}^d \to \mathbb{R}$ be integrable.
   (a) For any $y \in \mathbb{R}^d$ show that $\int_{\mathbb{R}^d} f(x + y) \, d\lambda(x) = \int_{\mathbb{R}^d} f(x) \, d\lambda(x)$.
   (b) If $T : \mathbb{R}^d \to \mathbb{R}^d$ an invertible linear transformation, and $E \in \mathcal{L}(\mathbb{R}^d)$. Show that
$$\int_{T^{-1}(E)} (f \circ T) |\det T| \, d\lambda = \int_E f \, d\lambda.$$

Details in class I left for you to check.

* Check that if s, t are non-negative simple functions then $\int_X (s + t) = \int_X s + \int_X t$.
* Show that there exists $f : \mathbb{R} \to [0, \infty)$ Borel measurable such that $\int_a^b f \, d\lambda = \infty$ for all $a, b \in \mathbb{R}$ with $a < b \in \mathbb{R}$. [Hint: Let $g(x) = \chi_{\{|x|<1\}} |x|^{-1/2}$, and define $h(x) = \sum_{m=-\infty}^\infty \sum_{n=1}^\infty 2^{-m-n} g(x - m/n)$.]

Assignment 7: Assigned Wed 10/09. Due Wed 10/16

1. Do questions 3, 5, and 6 from HW6.
2. (a) *(Jensen's inequality)* Let $a, b \in [-\infty, \infty]$ with $a < b$ and $\varphi : (a, b) \to \mathbb{R}$ be a convex function. If $\mu(X) = 1$ and $f : X \to (a, b)$ is integrable then show

$$\varphi\Big(\int_X f\,d\mu\Big) \leqslant \int_X \varphi \circ f\,d\mu.$$

   (b) If φ above is strictly convex, when can you have equality?
3. (a) Suppose $p, q, r \in [1, \infty]$ with $p < q < r$. Prove that for all $f \in L^p \cap L^r$, $f \in L^q$. Further, find $\theta \in (0, 1)$ such that $\|f\|_q \leqslant \|f\|_p^{\theta} \|f\|_r^{1-\theta}$.
   (b) If for some $p \in [1, \infty)$, $f \in L^p(X) \cap L^\infty(X)$ show that $\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$. [This sort of justifies the notation $\|\cdot\|_\infty$.]
   (c) Let $p_0 \in (0, \infty]$, $\mu(X) = 1$ and $f \in L^{p_0}(X)$. Prove $\lim_{p\to 0^+} \|f\|_p = \exp\big(\int_X \ln|f|\,d\mu\big)$.

Optional problems, and details in class I left for you to check.

- Let $g \geqslant 0$ be measurable, and define $\nu(A) = \int_A g\,d\mu$. Show that ν is a measure, and $\int_E f\,d\nu = \int_E fg\,d\mu$.
- Prove Hölder's inequality for $p = 1$ and $q = \infty$.
- If $p_i, q \in [1, \infty]$ with $\sum_1^N \frac{1}{p_i} = \frac{1}{q}$, show that $\|\prod_1^n f_i\|_q \leqslant \prod \|f_i\|_{p_i}$.
- Show that L^∞ is a Banach space.
- For $p \in [0, 1)$ show that you need not have $\|f + g\|_p \leqslant \|f\|_p + \|g\|_p$.
- Let $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p$ and $g \in L^q$. Show that $\int_X |fg|\,d\mu = \|f\|_p \|g\|_q$ if and only if there exists constants $\alpha, \beta \geqslant 0$ such that $\alpha f^p = \beta g^q$.
- (a) If X is σ-finite, then show $\|f\|_\infty = \sup_{g \in L^1 - \{0\}} \frac{1}{\|g\|_1} \int_X fg\,d\mu$.
  (b) Show that the previous subpart is false if X is not σ-finite.

Assignment 8: Assigned Wed 10/16. Due Wed 10/23

1. (a) If $\mu(X) < \infty$, $1 \leqslant p < q \leqslant \infty$, show $L^q(X) \subseteq L^p(X)$ and the inclusion map from $L^q(X) \to L^p(X)$ is continuous. Find an example where $L^q(X) \subsetneq L^p(X)$. [Hint: Show $\|f\|_p \leqslant \mu(X)^{\frac{1}{p}-\frac{1}{q}} \|f\|_q$.]
   (b) Let $\ell^p = L^p(\mathbb{N})$ with respect to the counting measure. If $1 \leqslant p < q$ show that $\ell^p \subsetneq \ell^q$. Is the inclusion map $\ell^p \hookrightarrow \ell^q$ continuous? Prove your answer.
2. (a) Suppose $p \in [1, \infty)$, and $f \in L^p(\mathbb{R}^d, \lambda)$. For $y \in \mathbb{R}^d$, let $\tau_y f : \mathbb{R}^d \to \mathbb{R}$ be defined by $\tau_y f(x) = f(x - y)$. Show that $(\tau_y f) \to f$ in L^p as $|y| \to 0$.
   (b) What happens for $p = \infty$?
3. Suppose $\Sigma = \sigma(\mathcal{C})$, where $\mathcal{C} \subseteq \mathcal{P}(X)$ is countable. If μ is a σ-finite measure and $1 \leqslant p < \infty$, show that $L^p(X)$ is seperable (i.e. has a countable dense subset).
4. (a) Suppose $\lim_{\lambda\to\infty} \sup_n \int_{|f_n|>\lambda} |f_n|\,d\mu = 0$. Show that there exists an increasing funciton φ with $\varphi(\lambda)/\lambda \to \infty$ as $\lambda \to \infty$, such that $\sup_n \int_X \varphi(|f_n|) < \infty$.
   (b) Suppose $\{f_n\}$ is uniformly integrable, and $\sup_n \int |f_n| < \infty$. Show that $\lim_{\lambda\to\infty} \sup_n \int_{|f_n|>\lambda} |f_n| = 0$.
   (c) Show that the previous part fails without the assumption $\sup_n \int |f_n| < \infty$.
5. Let $e_n(x) = e^{2\pi i n x}$, $X = [0, 1]$. For what $p \in [1, \infty]$ does $\{e_n\}$ have a convergent subsequence in $L^p(X, \lambda)$? Prove it.

Optional problems, and details in class I left for you to check.

- In Vitali's convergence theorem prove that the assumption $f \in L^1$ is unnecessary.
- If $(f_n) \to f$ in L^1, show that $\{f_n\}$ is uniformly integrable. [This is part of Vitali's theorem which I didn't have time to prove in class.]
- Show that if $(f_n) \to f$ in measure, then (f_n) need not converge to f in L^p.
- Finish the proof that $C_c(X)$ is dense in L^p. [I only did the case when X is compact in class.]
- Show that simple functions are dense in L^∞.
- Show that $C_c(\mathbb{R})$ is not dense in $L^\infty(\mathbb{R})$.
- Show that $L^\infty(\mathbb{R})$ is not separable.

Assignment 9: Assigned Wed 10/23. Due Wed 10/30

1. Recall we defined the variation of μ by $|\mu| = \mu^+ + \mu^-$, and the total variation by $\|\mu\| = |\mu|(X)$. (You should check that these are well defined.)
   (a) Let $\mathcal{M}$ be the space of all finite signed measures on (X, Σ). Show that $\mathcal{M}$ with total variation norm (i.e. with $\|\mu\| = |\mu|(X)$) is a Banach space.
   (b) Show that $(\mu_n) \to \mu$ if and only if $(\mu_n(A)) \to \mu(A)$ uniformly in A, $\forall A \in \Sigma$.
2. (a) For a signed measure, we define $\int_X f\,d\mu = \int_X f\,d\mu^+ - \int_X f\,d\mu^-$. Suppose $(f_n) \to f$, $(g_n) \to g$, and $|f_n| \leqslant g_n$ almost everywhere with respect to $|\mu|$. If $\lim \int_X g_n\,d|\mu| = \int_X g\,d|\mu| < \infty$, show that $\lim \int_X f_n\,d\mu = \int_X f\,d\mu$.
   (b) Suppose $f, f_n \in L^1$, and $(f_n) \to f$ almost everywhere. Show that $\lim \int |f_n - f| d|\mu| = 0$ if and only if $\lim \int |f_n|\,d|\mu| = \int |f| d|\mu|$.
3. (a) If μ is a positive σ-finite measure, and ν is a finite signed measure such that $|\nu| \ll \mu$, show that there exists $f \in L^1(X, \mu)$ such that $d\nu = f\,d\mu$.
   (b) Compute $\frac{d\nu}{d|\nu|}$ in terms of the Hanh decomposition of ν. [NOTATION: We say $g = \frac{d\nu}{d\mu}$ if $d\nu = g\,d\mu$.]
4. (a) Let ν_1 and ν_2 be two finite signed measures on X. Show that there exists a finite signed measure $\nu_1 \vee \nu_2$ such that $\nu_1 \vee \nu_2(A) \geqslant \nu_1(A) \vee \nu_2(A)$, and for any other finite signed measure ν such that $\nu(A) \geqslant \nu_1(A) \vee \nu_2(A)$ we ust have $\nu_1 \vee \nu_2 \leqslant \nu$.
   (b) If ν_1, ν_2 above are absolutely continuous with respect to a positive σ-finite measure μ, prove $\nu_1 \vee \nu_2 \ll \mu$ and express $\frac{d(\nu_1 \vee \nu_2)}{d\mu}$ in terms of $\frac{d\nu_1}{d\mu}$ and $\frac{d\nu_2}{d\mu}$.
5. Let $(\Omega, \mathcal{F}, P)$ be a measure space with $P(\Omega) = 1$, and $X \in L^1(\Omega, \mathcal{F}, P)$. [The probabilistic interpretation is that Ω is the sample space, $A \in \mathcal{F}$ is an event, X is a random variable, and $P(X \in B)$ is the chance that $X \in B$, where $B \in \mathcal{B}(\mathbb{R})$.]
   (a) Suppose $\mathcal{G} \subseteq \mathcal{F}$ is a σ-sub-algebra of F. Show that there exists a unique $\mathcal{G}$-measurable function Y such that $\int_A Y\,dP = \int_A X\,dP$ for all $A \in \mathcal{G}$. [Y is called the *conditional expectation* of X given $\mathcal{G}$, and denoted by $E(X\,|\,\mathcal{G})$.]
   (b) *(Tower property)* If $\mathcal{H} \subseteq \mathcal{G}$ is a σ-sub-algebra, show that $E(X\,|\,\mathcal{H}) = E\big(E(X\,|\,\mathcal{G})\,|\,\mathcal{H}\big)$ almost everywhere.
   (c) *(Conditional Jensen)* If $\varphi : \mathbb{R} \to \mathbb{R}$ is convex, show that $\varphi(E(X\,|\,\mathcal{G})) \leqslant E(\varphi(X)\,|\,G)$ almost everywhere.
   (d) Suppose $X \in L^2(\Omega, \mathcal{F}, P)$. Show that $E(X\,|\,\mathcal{G})$ is the L^2-orthogonal projection of X onto the subspace $L^2(\Omega, \mathcal{G})$. [Namely show $E(X\,|\,\mathcal{G}) \in L^2(\Omega, \mathcal{G})$, and $\int_\Omega (X - E(X\,|\,G))Y\,dP = 0$ for all $Y \in L^2(\Omega, \mathcal{G})$.]

Optional problems, and details in class I left for you to check.

* In the proof of the Hanh decomposition, prove the following: Say $\mu(X) > -\infty$, and $\alpha = \inf\{\mu(B)\}$. Let B'_n be a sequence of negative sets such that $\mu(B'_n) \to \alpha$. Let $N = \cup B'_n$. Show $\mu(N) = \alpha$.
* Prove the Hanh decomposition is unique up to null sets.
* Prove uniqueness of the Jordan decomposition.
* Show that the Radon-Nikodym theorem need not hold if μ, ν are not σ-finite.

Assignment 10: Assigned Wed 10/30. Due Wed 11/06

1. (a) Let $p \in [1, \infty)$ and q be conjugate Hölder exponent. If X is σ-finite, show that there exists a bijective linear isometry between $(L^p)^*$ and L^q.
   (b) The above result is *false* for $p = \infty$ even when $\mu(X) < \infty$. Find where our proof from class (when $\mu(X) < \infty$) fails when $p = \infty$.
   (c) We can (partially) construct a counter example on ℓ^∞ as follows. The Hanh-Banach theorem shows that there exists exists $T \in (\ell^\infty)^*$ such that $Ta = \lim a_n$, for all $a = (a_n) \in \ell^\infty$ such that $\lim a_n$ exists and is finite. Show that there does not exist $b \in \ell^1$ such that $Ta = \sum a_n b_n$ for all $a \in \ell^\infty$.
2. (a) Suppose $\sum_{m=1}^\infty (\sum_{n=1}^\infty |a_{m,n}|) < \infty$. Show that $\sum_{m=1}^\infty (\sum_{n=1}^\infty a_{m,n}) = \sum_{n=1}^\infty (\sum_{m=1}^\infty a_{m,n})$.
   (b) Give a counter example to (a) if we only assume $\sum_m \sum_n a_{m,n} < \infty$. Find a counter example where both iterated sums are finite.
3. (a) If X and Y are not σ-finite, show that Fubini's theorem need not hold.
   (b) If $\int_{[-1,1]^2} f\,d\lambda$ is not assumed to exist (in the extended sense), show that both iterated integrals can exist, be finite, but need not be equal.
4. *(Fubini for completions.)* Suppose (X, Σ, μ) and (Y, τ, ν) are two σ-finite, complete measure spaces. Let $\varpi = (\Sigma \otimes \tau)_\pi$ denote the completion of $\Sigma \otimes \tau$ with respect to the product measure $\pi = \mu \times \nu$.
   (a) Show that $\Sigma \otimes \tau$ need not be π-complete (i.e. $\varpi \supsetneq \Sigma \otimes \tau$ in general).
   (b) Suppose $f : X \times Y \to [-\infty, \infty]$ is $\mathcal{F}$-measurable. Define as usual the slices $\varphi_{f,x} : Y \to [0, \infty]$ by $\varphi_{f,x}(y) = f(x, y)$, and similarly $\psi_{f,y}(x) = f(x, y)$. Show that for μ-almost all $x \in X$, $\varphi_{f,x}$ is an τ-measurable, and for ν-almost all y, $\psi_{f,y}$ is an Σ-measurable.
   (c) Suppose f is integrable on $X \times Y$ in the extended sense. Define $F(x) = \int_Y f(x, y)\,d\nu(y)$ and $G(y) = \int_X f(x, y)\,d\mu(x)$. Show F is defined μ-a.e. and Σ-measurable. Similarly show G is defined ν-a.e., and τ-measurable. Further, show and that $\int_X F\,d\mu = \int_Y G\,d\nu = \int_{X \times Y} f d(\mu \times \nu)$.
5. Let (X, Σ, μ), (Y, τ, ν) be two σ-finite measure spaces, $p \in [1, \infty]$, and $f : X \times Y \to \mathbb{R}$ is $\Sigma \otimes \tau$ measurable. Let $F(x) = \int_Y f(x, y)\,d\nu(y)$, and $\psi_{y,f}$ be the slice of f defined by $\psi_{y,f}(x) = f(x, y)$. Show that $\|F\|_{L^p(X)} \leqslant \int_Y \|\psi_{y,f}\|_{L^p(X)}\,d\nu(y)$. [When $Y = \{1, 2\}$ with the counting measure, this is exactly Minkowski's triangle inequality.]

Optional problems, and details in class I left for you to check.

* Let $\mu(X) < \infty$, $p \in [1, \infty)$ and $T \in (L^p)^*$. Let $\nu(A) = T(\chi_A)$. We've seen in class that $\nu \ll \mu$ and so $d\nu = g\,d\mu$ for some $g \in L^1(\mu)$.
  (a) Show that $Tf = \int_X fg\,d\mu$ for all f simple.
  (b) If $\frac{1}{p} + \frac{1}{q} = 1$ show $\|g\|_q = \sup\{\int_X sg\}$, where the supremum runs over all *simple* functions s such that $\|s\|_p \leqslant 1$. Conclude $g \in L^q$ and $\|g\|_q \leqslant \|T\|$.
  (c) Show that $Tf = \int_X fg\,d\mu$ for all $f \in L^p$, to conclude the proof.
* Show that the Lebesgue measure on $\mathbb{R}^{m+n}$ is the product of the Lebesgue measures on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

Assignment 11: Assigned Wed 11/06. Due Wed 11/13

1. If $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p$, $g \in L^q$ show that $f * g$ is bounded and continuous. If $p, q < \infty$, show further $f * g(x) \to 0$ as $|x| \to \infty$.
2. Let $\{\varphi_n\}$ be an approximate identity.
   (a) If $f \in C(\mathbb{R}^d) \cap L^\infty$, show $f * \varphi_n \to f$ pointwise.
   (b) For $\alpha \in (0,1)$ define
   $$\|f\|_{C^\alpha} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}, \text{ and } C^\alpha = \{f : \mathbb{R}^d \to \mathbb{R} \mid \|f\|_{C^\alpha} < \infty\}.$$
   If $f \in C^\alpha$, show that $f * \varphi_n \in C^\alpha$ ~~and $f * \varphi_n \to f$ in C^α~~.
3. Define $\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \mid \forall m, \alpha,\ \sup_x (1 + |x|^m)|D^\alpha f(x)| < \infty\}$. Here $m \in \mathbb{N} \cup \{0\}$, and $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$ is a multi-index, and $D^\alpha f = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f$. The space $\mathcal{S}$ is called the *Schwartz Space.*
   (a) If $p \in [1, \infty)$, $f \in L^p(\mathbb{R}^d)$, $g \in \mathcal{S}(\mathbb{R}^d)$, show that $f * g \in C^\infty(\mathbb{R}^d)$, and further $D^\alpha(f * g) = f * (D^\alpha g)$.
   (b) For $p \in [1, \infty)$, show that C_c^∞ and $\mathcal{S}$ are dense subsets of L^p
4. Let $A, B \in \mathcal{L}(\mathbb{R})$ be measurable, and define $A + B = \{a + b \mid a \in A, b \in B\}$. If $\lambda(A) > 0$ and $\lambda(B) > 0$ show $A + B$ contains an interval.

Though I encourage you to check the properties on the Dirichlet and Fejér kernels stated in the optional problems, you may assume them here without proof.

Let $C_{\text{per}} = \{f \in C(\mathbb{R}) \mid \tau_1 f = f\}$ denote all continuous functions with period 1. Since the Fejér kernels are an approximate identity, it immediately follows that the Cesàro sums $\sigma_N f \to f$ uniformly, for any $f \in C_{\text{per}}$. For general $f \in C_{\text{per}}$, however, the partial sums $S_N f$ need not converge to f even pointwise. (In fact, there exist many $f \in C_{\text{per}}$ such that $S_N f$ is divergent on a dense G_δ.) If, however, f is a little bit better than continuous, then the Fourier series of f converges to f pointwise.

5. Let $\alpha \in (0,1)$ and $f \in C^\alpha_{\text{per}}$. Show that $(S_N f) \to f$ uniformly, as $N \to \infty$.

Optional problems, and details in class I left for you to check.

* If $f \in L^p$, $g \in L^q$ with $p, q \in [1, \infty]$ and $1/p + 1/q \geqslant 1$, show that $f * g = g * f$.
* If $f \in L^p$, $g \in L^q$, $h \in L^r$ with $p, q, r \in [1, \infty]$ and $1/p + 1/q + 1/r \geqslant 2$, show that $(f * g) * h = f * (g * h)$.
* Define the Derichlet kernel by $D_N(x) = \sum_{-N}^N \exp(2\pi i n x)$.
  (a) Show that $S_N f(x) = D_N * f(x) \overset{\text{def}}{=} \int_0^1 f(y) D_N(x - y)\, dy$. [Recall, $S_N f = \sum_{-N}^N \hat{f}(n) e_n$, where $e_n(x) = e^{2\pi i n x}$, and $\hat{f}(n) = \langle f, e_n \rangle = \int_0^1 f(y) \bar{e}_n(y)\, dy$.]
  (b) Show that $D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}$. Further show $\lim_{N \to \infty} \int_\varepsilon^{1-\varepsilon} |D_N| = \infty$.
* Define Fejér kernel by $F_N = \frac{1}{N} \sum_0^{N-1} D_n$.
  (a) Show that $\sigma_N f \overset{\text{def}}{=} \frac{1}{N} \sum_0^{N-1} S_n f = F_N * f$.
  (b) Show that $F_N(x) = \frac{\sin^2(N\pi x)}{N \sin^2(\pi x)}$, and that $\{F_N\}$ is an approximate identity.

Assignment 12: Assigned Wed 11/13. Due Wed 11/20

1. Let μ be a finite signed Borel measure on $[0,1]$. If $\forall n \in \mathbb{Z}\ \hat{\mu}(n) = 0$, show $\mu = 0$.
2. Let $0 \leqslant r < s$. Show that any bounded sequence in H^s_{per} has a subsequence that is convergent in H^r_{per}.
3. Let $f \in L^2([0,1])$. Show that there exists a unique $u \in C^\infty(\mathbb{R} \times (0, \infty))$ such that $u(x+1, t) = u(x, t)$, $\lim_{t \to 0^+} \|u(\cdot, t) - f(\cdot)\|_{L^2_{\text{per}}} = 0$, and $\partial_t u - \partial_x^2 u = 0$. [Hint: You may assume the result of the optional problems.]
4. Let $s \in (0,1]$ and $f \in L^2_{\text{per}}$. Prove $f \in H^s_{\text{per}} \iff \sup_{0 < h \leqslant 1} h^{-\alpha} \|\tau_h f - f\|_{L^2} < \infty$.
   [Update: The converse is false. A correction with solution will be posted.]
5. (a) Let $n \in \mathbb{N}$ be even, $\frac{1}{n} + \frac{1}{n'} = 1$. If $\hat{f} \in \ell^{n'}(\mathbb{Z})$, show that $f \in L^n_{\text{per}}([0,1])$ and $\|f\|_{L^n} \leqslant \|\hat{f}\|_{\ell^{n'}}$. [Hint: Let $n = 2m$. Then $\|f\|_{L^n}^n = \|(f^m)^\wedge\|_{\ell^2}^2$.]
   (b) Let $s > \frac{1}{2} - \frac{1}{p} \geqslant 0$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in H^s_{\text{per}}$ show $\hat{f} \in \ell^q(\mathbb{Z})$. Further show that the map $f \mapsto \hat{f}$ is continuous from $H^s_{\text{per}} \to \ell^q$.
   (c) If $n \in \mathbb{N}$ is even, $s > \frac{1}{2} - \frac{1}{n}$ then show that $H^s_{\text{per}} \subseteq L^n([0,1])$ and that the inclusion map is continuous. [This is one of the Sobolev embedding theorems.]

Optional problems, and details in class I left for you to check.

* (a) If $f, g \in L^2_{\text{per}}([0,1])$, show that $(f * g)^\wedge(n) = \hat{f}(n)\hat{g}(n)$.
  (b) If $f, g \in L^2_{\text{per}}([0,1])$, show that $(fg)^\wedge(n) = \hat{f} * \hat{g}(n) \overset{\text{def}}{=} \sum_{m \in \mathbb{Z}} \hat{f}(m)\hat{g}(n-m)$.
* (a) If $\alpha \in (0,1)$, $f \in C^\alpha_{\text{per}}([0,1])$, show that $\lim_{|n| \to \infty} |n|^\alpha |\hat{f}(n)| = 0$.
  (b) Show by example that the converse of the previous part is false.
* For any $s \geqslant 0$ show that H^s_{per} is a closed subspace of L^2.
* Let $s \geqslant 1$, and $f \in H^s_{\text{per}}$. Show that f has a weak derivative Df, and $Df \in H^{s-1}$. Further, show that the map $f \mapsto Df : H^s \to H^{s-1}$ is linear and continuous.
* Let $s > 3/2$ and $f, g \in H^s_{\text{per}}$. Show that $fg \in H^1$, and further $D(fg) = (Df)g + f(Dg)$.
* Find a function $f \in H^{1/2}_{\text{per}} - L^\infty$. [So the Sobolev embedding theorem is false for $s = 1/2$.]
* Let $n \in \mathbb{N} \cup \{0\}$, $\alpha \in [0,1)$ $s > 1/2 + n + \alpha$. Show that $H^s_{\text{per}} \subseteq C^{n,\alpha}_{\text{per}}[0,1]$ and the inclusion map is continuous. [Recall $C^{n,\alpha}_{\text{per}}[0,1]$ is the set of all C^n periodic functions on $\mathbb{R}$ (i.e. $\tau_1 f = f$) whose n^{th} derivative is Hölder continuous with exponent α.]
* Show that $f \in H^s$ for all $s \geqslant 0 \iff f \in C^\infty_{\text{per}}$.

Assignment 13: Assigned Wed 11/20. Due Wed 11/27

1. Let $s > 3/2$ and $f, g \in H^s_{\text{per}}$. Show that $fg \in H^1_{\text{per}}$, and further $D(fg) = (Df)g + f(Dg)$. [This was optional on last times homework.]
2. (a) If $f \in L^1(\mathbb{R}^d)$ and f is not identically 0 (a.e.), then show that $Mf \notin L^1(\mathbb{R}^d)$.

   The next few subparts outline a proof that for any $p > 1$, the maximal function is an L^p bounded sublinear operator. Let $p \in (1, \infty)$, $f \in L^p(\mathbb{R}^d)$ and $f \geqslant 0$.

   (b) Show that $\lambda\{Mf > \alpha\} \leqslant \frac{3^d}{(1-\delta)\alpha} \int_{\{f > \delta\alpha\}} f$, for any $t > 0$, $\delta \in (0, 1)$ and $f \geqslant 0$ measurable.

   (c) Let $p \in (1, \infty]$, and $d \in \mathbb{N}$. Show that there exists a constant $c = c(p, d)$ such that $\|Mf\|_p \leqslant c\|f\|_p$ for all $f \in L^p(\mathbb{R}^d)$. [Hint: For $p < \infty$, use the previous part, the identity $\|Mf\|_p^p = \int_0^\infty p\alpha^{p-1}\lambda\{Mf > \alpha\}\, d\alpha$ and optimise in δ.]
3. Let μ be a finite signed Borel measure on $\mathbb{R}^d$ such that $\mu \perp \lambda$. Show that $D|\mu| = \infty$, μ-almost everywhere.
4. Let $\alpha \in [0, d]$, and $A \in \mathcal{B}(\mathbb{R}^d)$. If $H_\alpha(A) < \infty$, show $\lim\limits_{r \to 0} \frac{H_\alpha(A \cap B(x,r))}{c_\alpha r^\alpha} = 0$ for H_α-almost all $x \notin A$.
5. (a) Suppose $f : [a, b] \to \mathbb{R}$ is a right continuous increasing function. Show that there exists a finite Borel measure μ such that $\mu((x, y]) = f(y) - f(x)$ for every $x, y \in [a, b]$. Show further that $\mu = \mu_{\text{ac}} + \mu_{\text{s}} + \sum_i \alpha_i \delta_{a_i}$, where $\mu_{\text{ac}} \ll \lambda$, $\alpha_i > 0$, $a_i \in [a, b)$, $\sum_i \alpha_i < \infty$, and $\mu_{\text{sc}} \perp \lambda$ is such that $\mu_{\text{sc}}(\{x\}) = 0$ for all $x \in \mathbb{R}$. [Hint: If f is strictly increasing and continuous, define $\mu(A) = \lambda(f(A))$, and consider its Lebesgue decomposition.]

   (b) Let $f : [a, b] \to \mathbb{R}$ be monotone. Show that f is differentiable almost everywhere, $f' \in L^1([a, b])$ and that $|\int_a^b f'| \leqslant |f(b) - f(a)|$.

Optional problems, and details in class I left for you to check.

* Let $c_\alpha = \frac{\pi^{\alpha/2}}{\Gamma(1+\frac{\alpha}{2})}$ be the normalization constant from the definition of H_α, the Hausdorff measure of dimension α.

  (a) If $0 < H_\alpha(A) < \infty$, show $\limsup\limits_{r \to 0} \frac{H_\alpha(A \cap B(x,r))}{c_\alpha r^\alpha} \in [2^{-\alpha}, 1]$ for H_α-a.e. $x \in A$.

  (b) Show that there exists $\alpha < d$ and $A \subseteq \mathbb{R}^d$ with $H_\alpha(A) \in (0, \infty)$ such that

  $$\liminf_{r \to 0} \frac{H_\alpha(A \cap B(x,r))}{c_\alpha r^\alpha} = 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{H_\alpha(A \cap B(x,r))}{c_\alpha r^\alpha} < 1,$$

  for H^α-almost every $x \in \mathbb{R}^d$.

  (c) If C is the Cantor set, and $\alpha = \log 2/\log 3$, compute $\limsup\limits_{r \to 0} \frac{H_\alpha(C \cap B(x,r))}{c_\alpha r^\alpha}$.
* *(Infinite version of Vitali.)* Suppose $A \subseteq \cup B_\alpha$, where $\{B_\alpha\}_{\alpha \in \mathcal{A}}$ is an infinite collection of balls such that $\sup \lambda(B_\alpha) < \infty$. Show that there exists $\mathcal{A}' \subseteq \mathcal{A}$ such that the sub-collection $\{B_{\alpha'}\}_{\alpha' \in \mathcal{A}'}$ is disjoint and $A \subseteq \cup 5B_{\alpha'}$.
* If $f \in L^1(\mathbb{R}^d)$, show that $Mf(x) \geqslant |f(x)|$ at all Lebesgue points of f.

Assignment 14: Assigned Wed 11/27. Due Wed 12/04

1. Let μ be a positive finite Borel measure on $\mathbb{R}^d$, and $\alpha > 0$. Show that for every $A \subseteq \{D\mu > \alpha\}$, we must have $\mu(A) \geqslant \alpha\lambda(A)$.
2. (a) *(Polar Coordinates.)* Let $f \in L^1(\mathbb{R}^2)$. Show that

   $$\int_{\mathbb{R}^2} f(x, y)\, dx\, dy = \int_{[0,\infty)\times[0,2\pi)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

   (b) *(Higher dimensional version.)* Let $f \in L^1(\mathbb{R}^d)$. Let $S_1 = \{y \in \mathbb{R}^d \mid |y| = 1\}$ be the $d-1$ dimensional sphere of radius 1. Show that there exists a unique measure σ on S_1 such that

   $$\int_{\mathbb{R}^d} f(x)\, dx = \int_{r\in[0,\infty)} \int_{y \in S_1} f(ry)\, r^{d-1}\, d\sigma(y)\, d\lambda(r).$$

   [Hint: For $A \in \mathcal{B}(S_1)$ define $\sigma(A) = \lambda(A^*)$ where $A^* = \{rx \mid x \in A, r \in [0, 1]\}$. Now for any $B \in \mathcal{B}(S_1)$ prove the desired equality when $f = \chi_A$ where $A = \{rx \mid a < r < b, x \in B\}$.]

Optional problems, and details in class I left for you to check.

* Show that the arbitrary union of closed (non-degenerate) cells is Lebesgue measurable.
* Find an example of $E \in \mathcal{L}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ such that $\lim_{r\to 0} \frac{\lambda(E \cap B(x,r))}{\lambda(B(x,r))}$ does not exist.
* Suppose $f : \mathbb{R} \to \mathbb{R}$ is measurable. Let $\alpha, \beta > 0$ with $\alpha/\beta \notin \mathbb{Q}$. If f has period α, and also has period β (i.e. for all $x \in \mathbb{R}$, $f(x) = f(x + \alpha) = f(x + \beta)$), then show that f is constant almost everywhere. (But f need not be constant everywhere!)
* We say the family $\{E_r\}$ *shrinks nicely* to $x \in \mathbb{R}^d$ if there exists $\delta > 0$ such that for all r, $E_r \subseteq B(x, r)$ and $\lambda(E_r) > \delta\lambda(B(x, r))$. If $\{E_r\}$ shrinks nicely to x, show that $\lim \frac{1}{\lambda(E_r)} \int_{E_r} f = f(x)$ for all Lebesgue points of f.
* If $f \in L^1(\mathbb{R}^d)$, show that $Mf(x) \geqslant |f(x)|$ at all Lebesgue points of f.
* If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then show that f is of bounded variation, and that the variation is absolutely continuous. Conclude f can be written as the difference of two monotone absolutely continuous functions.
* Let $U, V \subseteq \mathbb{R}^d$ be open and $\varphi : U \to V$ be C^1 and injective. If $x_0 \in U$ and $\nabla\varphi(x_0)$ is not invertible, show that

  $$\lim_{r\to 0} \frac{\lambda(\varphi(B(x_0, r)))}{\lambda(B(x_0, r))} = 0.$$