

Assignment 15: Assigned Wed 12/04. Due never

1. (a) If $\int_{\mathbb{R}^d} (1 + |x|)|f(x)| dx < \infty$, show that \hat{f} is differentiable and $\partial_j \hat{f}(\xi) = -2\pi i(x_j f(x))^\wedge(\xi)$. [NOTE: $(x_j f(x))^\wedge(\xi)$ means $\hat{g}(\xi)$, where $g(x) = x_j f(x)$.]
 (b) If $f \in C_0^1(\mathbb{R}^d)$, and $\nabla f \in L^1$ show that $(\partial_j f)^\wedge(\xi) = +2\pi i \xi_j \hat{f}(\xi)$.
 (c) Show that the mapping $f \mapsto \hat{f}$ is a bijection in the Schwartz space.
2. If μ is a finite Borel measure on \mathbb{R}^d define $\hat{\mu}(\xi) = \int e^{-2\pi i(x,\xi)} d\mu(x)$. If $\hat{\mu}(\xi) = 0$ for all ξ , show that $\mu = 0$. [HINT: Show that $\int f d\mu = 0$ for all $f \in \mathcal{S}$.]
3. For $f \in L^1$, the formula $\hat{f}(\xi) = \int f(x)e^{-2\pi i(x,\xi)}$ allows us to prove many identities: E.g. $(\delta_\lambda f)^\wedge(\xi) = \hat{f}(\lambda\xi)$, etc. For $f \in L^2$, the formula $\hat{f}(\xi) = \int f(x)e^{-2\pi i(x,\xi)}$ is no longer valid, as the integral is not defined (in the Lebesgue sense). However, most identities remain valid, and can be proved using an approximation argument. I list a couple here.
 (a) For $f \in L^1$ we know $(\tau_x f)^\wedge(\xi) = e^{-2\pi i(x,\xi)} \hat{f}(\xi)$. Prove it for $f \in L^2$.
 (b) Similarly, show that $(\delta_\lambda f)^\wedge(\xi) = \hat{f}(\lambda\xi)$ for all $f \in L^2$.
 (c) Let F denote the Fourier transform operator (i.e. $Ff = \hat{f}$), and R denote the reflection operator (i.e. $Rf(x) = f(-x)$). Note that our Fourier inversion formula (for $f \in L^1, \hat{f} \in L^1$) is exactly equivalent to saying $F^2 f = Rf$. Prove $F^2 f = Rf$ for all $f \in L^2$.

4. (*Uncertainty principle*) Suppose $f \in \mathcal{S}(\mathbb{R})$. Show that

$$\left(\int_{\mathbb{R}} |xf(x)|^2 dx \right) \left(\int_{\mathbb{R}} |\xi \hat{f}(\xi)|^2 d\xi \right) \geq \frac{1}{16\pi^2} \|f\|_{L^2}^2 \|\hat{f}\|_{L^2}^2$$

[This illustrates a nice localisation principle about the Fourier transform. The first integral measures the spread of the function f . The second the spread of the Fourier transform \hat{f} . Here you show that this product is bounded below! The proof, once you know enough Physics, reduces to the above inequality.

Hint: Consider $\int_{\mathbb{R}} xf(x)f'(x) dx$.

5. (*Central limit theorem*) Let $f \in L^1(\mathbb{R})$ be such that $f \geq 0$ and $\int x^2 f(x) dx < \infty$. Define $g_n = (f * \dots * f)$ (n -times), and $h_n(x) = \delta_{1/\sqrt{n}} g_n(x) = \sqrt{n} g_n(\sqrt{n}x)$. Show

$$\hat{h}_n(\xi) \xrightarrow{n \rightarrow \infty} \exp(-2\pi i \mu \xi - 2\pi^2 i \sigma^2 \xi^2),$$

where $\mu = \int xf(x) dx$ and $\sigma^2 = \int (x - \mu)^2 f(x) dx$. [The central limit theorem says that tabulating results of a large number of independent trials of any experiment produces a “bell curve”. The key step in the proof, which you will no doubt see next semester, is showing that any function convolved with itself often enough looks like a Gaussian.]

6. (*Sobolev spaces*) For $f \in L^2(\mathbb{R}^d)$ and $s \geq 0$ define

$$\|f\|_{H^s}^2 = \int (1 + |\xi|^s)^2 |\hat{f}(\xi)|^2 d\xi, \quad \text{and} \quad H^s = \{f \in L^2 \mid \|f\|_{H^s} < \infty\}.$$

Intuitively, we think of H^s as the space of functions with “ s ” “weak-derivatives” in L^2 . (This will be formalized in your functional analysis course.)

- (a) If $f \in C_0^n(\mathbb{R}^d)$ and $D^\alpha f \in L^2$ for all $|\alpha| < n$, then show that $f \in H^n(\mathbb{R}^d)$.
- (b) Let $s \in (0, 1)$ and $f \in L^2(\mathbb{R}^d)$. Show that $f \in H^s(\mathbb{R}^d)$ if and only if

$$\int_0^\infty \left(\frac{\|\tau_h f - f\|_{L^2}}{h^s} \right)^2 \frac{dh}{h} < \infty.$$

7. (*Sobolev embedding*) If $n \in \mathbb{N}$ and $f \in H^s(\mathbb{R}^d)$ for $s > n + \frac{d}{2}$ then show that $f \in C^n$, and further the inclusion map $H^s \rightarrow C^n$ is continuous.
8. (a) (*Elliptic regularity*) Let $Lu = \sum a_{ij} \partial_i \partial_j u + \sum b_i \partial_i u + cu$, where a_{ij}, b_i, c are constants. Suppose $\exists \lambda > 0$ such that $a_{ij} = a_{ji}$ and $|\sum a_{ij} \xi_i \xi_j| \geq \lambda |\xi|^2$ for all $\xi \in \mathbb{R}^n$ (this assumption is called ellipticity). If $f \in \mathcal{S}$ and $u, \partial_i u, \partial_i \partial_j u \in L^2 \cap C^0$ are such that $Lu = f$, show that $u \in C^\infty$. [To emphasize why this is surprising, choose for example $L = \Delta$. Then $\Delta u = f$ makes no mention of a mixed derivative $\partial_1 \partial_2 u$. Yet, all such mixed derivatives exist and are smooth. Hint: If $f \in H^s$ show that $u \in H^{s+2}$.]
 (b) Show by example that the previous subpart is false without the ellipticity assumption.
9. (*Trace theorems*) Let $p \in \mathbb{R}^m$ be fixed. Given $f : \mathbb{R}^{m+n} \rightarrow \mathbb{R}$ define $S_p f : \mathbb{R}^n \rightarrow \mathbb{R}$ by $S_p f(y) = f(p, y)$.
 (a) Let $s > m/2$, and $s' = s - m/2$. Show that there exists a constant c such that $\|S_p f\|_{H^{s'}(\mathbb{R}^n)} \leq c \|f\|_{H^s(\mathbb{R}^{m+n})}$.
 (b) Show that the section operator S_p extends to a continuous linear operator from $H^s(\mathbb{R}^{m+n})$ to $H^{s'}(\mathbb{R}^n)$. [Given an arbitrary L^2 function on \mathbb{R}^{m+n} it is of course impossible to restrict it to an m -dimensional hyper-plane. However, if your function has more than $n/2$ “Sobolev derivatives”, then you can make sense of this restriction, and the restriction still has $s - n/2$ “Sobolev derivatives”.]
10. (*Reliech Lemma*) Let $K \subseteq \mathbb{R}^d$ be compact, $0 \leq s_1 < s_2$, and suppose $\{f_n\}$ are a sequence of functions supported in K . If the sequence $\{f_n\}$ is bounded in H^{s_2} , then show that it has a convergent subsequence in H^{s_1} . [This is the generalization of the Arzella-Ascoli theorem in this context.]