Assignment 15: Assigned Wed 12/04. Due never

1. (a) If \( \int_{\mathbb{R}^d} (1 + |x|) |f(x)| \, dx < \infty \), show that \( \hat{f} \) is differentiable and \( \partial_{\xi_j} \hat{f}(\xi) = -2\pi i \langle x_j f(x) \rangle \hat{\delta}(\xi) \). [Note: \( \langle x_j f(x) \rangle \hat{\delta}(\xi) \) means \( \hat{\delta}(\xi) \), where \( g(x) = x_j f(x) \).]

(b) If \( f \in C^2_0(\mathbb{R}^d) \), and \( \nabla f \in L^1 \) show that \( (\partial_{\xi_j} \hat{f})(\xi) = +2\pi i \xi_j \hat{f}(\xi) \).

(c) Show that the mapping \( f \to \hat{f} \) is a bijection in the Schwartz space.

2. If \( \mu \) is a finite Borel measure on \( \mathbb{R}^d \) define \( \hat{\mu}(\xi) = \int e^{-2\pi i x \cdot \xi} \, d\mu(x) \). If \( \hat{\mu}(\xi) = 0 \) for all \( \xi \), show that \( \mu = 0 \). [Hint: Show that \( \int f(x) \, d\mu(x) = 0 \) for all \( f \in \mathcal{S} \).]

3. For \( f \in L^1 \), the formula \( \hat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} \, dx \) allows us to prove many identities: e.g. \( (\delta \lambda f)(\xi) = \hat{f}(\lambda \xi) \), etc. For \( f \in L^2 \), the formula \( \hat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} \) is no longer valid, as the integral is not defined (in the Lebesgue sense). However, most identities remain valid, and can be proved using an approximation argument. I list a couple here.

(a) For \( f \in L^1 \) we know \( \langle \tau_\lambda x f \rangle(\xi) = e^{-2\pi i x \cdot \xi} \hat{f}(\xi) \). Prove it for \( f \in L^2 \).

(b) Similarly, show that \( (\delta \lambda f)(\xi) = \hat{f}(\lambda \xi) \) for all \( \xi \in L^2 \).

(c) Let \( F \) denote the Fourier transform operator (i.e. \( F f = \hat{f} \)), and \( R \) denote the reflection operator (i.e. \( R f(x) = f(-x) \)). Note that our Fourier inversion formula (for \( f \in L^1 \), \( \hat{f} \in L^1 \)) is exactly equivalent to saying \( F^2 f = R f \).

4. \((Uncertainty principle)\) Suppose \( f \in \mathcal{S}(\mathbb{R}) \). Show that

\[
\left( \int_{\mathbb{R}} |xf(x)|^2 \, dx \right) \left( \int_{\mathbb{R}} |\xi \hat{f}(\xi)|^2 \, d\xi \right) \geq \frac{1}{10\pi^2} \left\| f \right\|^2 \left\| \hat{f} \right\|^2.
\]

[This illustrates a nice localisation principle about the Fourier transform. The first integral measures the spread of the function \( f \). The second spread of the Fourier transform \( \hat{f} \). Here you show that this product is bounded below! The proof, once you know enough Physics, reduces to the above inequality.

Hint: Consider \( \int_{\mathbb{R}^d} x f(x) f'(x) \, dx \).]

5. \((Central limit theorem)\) Let \( f \in L^1(\mathbb{R}) \) be such that \( f \geq 0 \) and \( \int x^2 f(x) \, dx < \infty \). Define \( g_n = (f * \cdots * f) \) \((n\)-times\), and \( h_n(x) = \delta_1/\sqrt{n} g_n(x) = \sqrt{n} \hat{g}_n(\sqrt{n} x) \). Show

\[
\hat{h}_n(\xi) \xrightarrow{n \to \infty} \exp(-2\pi i \xi \mu - 2\pi^2 i \sigma^2 \xi^2),
\]

where \( \mu = \int x f(x) \, dx \) and \( \sigma^2 = \int (x - \mu)^2 f(x) \, dx \). [The central limit theorem says that tabulating results of a large number of independent trials of any experiment produces a “bell curve.” The key step in the proof, which you will no doubt see next semester, is showing that any function convolved with itself often enough looks like a Gaussian.]

6. \((Sobolev spaces)\) For \( f \in L^2(\mathbb{R}^d) \) and \( s \geq 0 \) define

\[
\|f\|_{H^s}^2 = \int (1 + |\xi|^s)^2 |\hat{f}(\xi)|^2 \, d\xi,
\]

and \( H^s = \{ f \in L^2 \mid \|f\|_{H^s} < \infty \} \).

Intuitively, we think of \( H^s \) as the space of functions with “s” “weak-derivatives” in \( L^2 \). (This will be formalized in your functional analysis course.)

(a) If \( f \in C^0_0(\mathbb{R}^d) \) and \( D^\alpha f \in L^2 \) for all \( |\alpha| < n \), then show that \( f \in H^n(\mathbb{R}^d) \).

(b) Let \( s \in (0, 1) \) and \( f \in L^2(\mathbb{R}^d) \). Show that \( f \in H^s(\mathbb{R}^d) \) if and only if

\[
\int_0^\infty \left( \frac{\|\tau_\lambda f - f\|_{L^2}}{\lambda^s} \right)^2 \, d\lambda < \infty.
\]

7. \((Sobolev embedding)\) If \( n \in \mathbb{N} \) and \( f \in H^n(\mathbb{R}^d) \) for \( s > n + \frac{d}{2} \) then show that \( f \in C^n \), and further the inclusion map \( H^s \to C^n \) is continuous.

8. \((Elliptic regularity)\) Let \( Lu = \sum a_{ij} \partial_i \partial_j u + \sum b_i \partial_i u + cu \), where \( a_{ij}, b_i, c \) are constants. Suppose \( \exists \lambda > 0 \) such that \( a_{ij} = a_{ji} \) and \( \sum |a_{ij} \xi_i \xi_j| > \lambda |\xi|^2 \) for all \( \xi \in \mathbb{R}^n \) (this assumption is called ellipticity). If \( f \in F \) and \( u, \partial_i u, \partial_i \partial_j u \in L^2 \cap C^0 \) are such that \( Lu = f \), show that \( u \in C^\infty \). [To emphasize why this is surprising, choose for example \( L = \Delta \). Then \( \Delta u = f \) makes no mention of a mixed derivative \( \partial_i \partial_2 u \). Yet, all such mixed derivatives exist and are smooth. Hint: If \( f \in H^s \) show that \( u \in H^{s+2} \)].

(b) Show by example that the previous subpart is false without the ellipticity assumption.

9. \((Trace theorems)\) Let \( p \in \mathbb{R}^m \) be fixed. Given \( f : \mathbb{R}^{m+n} \to \mathbb{R} \) define \( S_p f : \mathbb{R}^n \to \mathbb{R} \) by \( S_p f(y) = f(p, y) \).

(a) Let \( s > m/2 \), and \( s' = s - m/2 \). Show that there exists a constant \( c \) such that \( \|S_p f\|_{H^{s'}(\mathbb{R}^n)} \leq c \|f\|_{H^s(\mathbb{R}^{m+n})} \).

(b) Show that the section operator \( S_p \) extends to a continuous linear operator from \( H^s(\mathbb{R}^{m+n}) \) to \( H^{s'}(\mathbb{R}^n) \). (Given an arbitrary \( L^2 \) function on \( \mathbb{R}^{m+n} \) it is of course impossible to restrict it to an \( m \)-dimensional hyper-plane. However, if your function has more than \( n/2 \) “Sobolev derivatives,” then you can make sense of this restriction, and the restriction still has \( s - n/2 \) “Sobolev derivatives.”]

10. \((Rellich Lemma)\) Let \( K \subseteq \mathbb{R}^d \) be compact, \( 0 \leq s_1 < s_2 \), and suppose \{\( f_n \)\} are a sequence of functions supported in \( K \). If the sequence \{\( f_n \)\} is bounded in \( H^{s_2} \), then show that it has a convergent subsequence in \( H^{s_1} \). [This is the generalization of the Arzelà-Ascoli theorem in this context.]