21-720 Measure Theory: Midterm.
December 10th, 2013

- This is a closed book test. No calculators or computational aids are allowed.
- You have 3 hours. The exam has a total of 8 questions and 35 points.
- You may use any result from class or homework PROVIDED it is independent of the problem you want to use the result in. (You must also CLEARLY state the result you are using.)
- The questions are ROUGHLY in order of length / difficulty, and not in the order the material was covered. However, depending on your intuition, you might find a few of the later questions easier. Good luck!

In this exam, we always assume $(X, \Sigma, \mu)$ is a measure space. We use $\lambda$ to denote the Lebesgue measure on $\mathbb{R}^d$.

1. Recall the maximal function of a function $f$ is defined by $Mf(x) = \sup_{r>0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f| \, d\lambda$. True or false: If $f \in L^1$, then $Mf$ is measurable. Prove it, or find a counter-example.

2. Consider the function $f(x,y) = \frac{xy}{x^2+y^4}$ when $(x,y) \neq 0$ and $f(0,0) = 0$ defined on the rectangle $[-1,1]^2$. Consider each of the following integrals:
   (a) $\int_{-1}^{1} \left( \int_{-1}^{1} f(x,y) \, d\lambda(x) \right) \, d\lambda(y)$
   (b) $\int_{-1}^{1} \left( \int_{-1}^{1} f(x,y) \, d\lambda(y) \right) \, d\lambda(x)$
   (c) $\int_{[-1,1]^2} f \, d\lambda$.
   For each of the integrals above, decide if they exist (in the extended sense) or not. If yes, compute it. If not, prove it.

3. Let $\phi_n$ be an approximate identity. True or false:
   For every $f \in L^1(\mathbb{R}^d)$ such that $\hat{f} \in L^1(\mathbb{R}^d)$ the sequence $(\phi_n \ast f)^\wedge$ is convergent in $L^1(\mathbb{R}^d)$. Prove, or find a counter example. [Recall that $(\phi_n)$ is an approximate identity if $\phi_n \geq 0$, $\int_{\mathbb{R}^d} \phi_n \, d\lambda = 1$, and for every $\varepsilon > 0$ we have $\lim_{n \to \infty} \int_{|x| > \varepsilon} \phi_n(x) \, dx = 0$.]

4. Let $g \in L^1(X,\mu)$, and $f : X \to \mathbb{R}$ be measurable. Let $(f_n)$ be a sequence of measurable functions such that $(f_n) \to f$ in measure, and for all $n \in \mathbb{N}$ we have $|f_n| \leq g$ almost everywhere. True or false:
   The sequence $(f_n)$ necessarily has a subsequence $(f_{n_k})$ such that $(f_{n_k}) \to f$ almost everywhere. Prove it, or find a counter example.

5. Let $f : [0,1] \to \mathbb{R}$ be absolutely continuous. Suppose further $f' \in L^2([0,1])$. True or false:
   For every $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon \sqrt{|x-y|}$. Prove it, or find a counter example.

6. True or false:
   For every $E \in \mathcal{L}(\mathbb{R})$ with $0 < \lambda(E) < \infty$, we must have $\lim_{n \to \infty} \int_E \cos^2(nx) \, dx$ exists. Prove it, or find a counter example.

7. Let $\varphi \in C_c(\mathbb{R}^d)$ be a non-negative radially decreasing function with $\int_{\mathbb{R}^d} \varphi = 1$. True or false:
   For all $f \in L^1(\mathbb{R}^d)$, we must have $f * \varphi \leq Mf$.
   Prove it, or find a counter example. [RECALL: We say $\varphi$ is radially decreasing if there exists a decreasing function $h : [0,\infty) \to \mathbb{R}$ such that $\varphi(x) = h(|x|)$]

If you've completed the remainder of this exam and have time to spare, here is a fun question. This is for your entertainment only, and will not influence your grade.

8. If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $f(x+y) = f(x) + f(y)$, then show $f$ is continuous.