Math 21-720: Measure Theory and Integration: Notes

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1 August 28, 2013

Let X be some set.

Definition 1.1 (σ-algebra)**.** Σ is a *σ-algebra* on X if

1. $\Sigma \subseteq \mathcal{P}(X)$,
2. $\emptyset \in \Sigma$,
3. $A \in \Sigma \Rightarrow A^c \in \Sigma$,
4. $A_i \in \Sigma$ for $i \in \mathbb{N} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \Sigma$.

Definition 1.2 (Positive measure)**.** μ is a *positive measure* on (X, Σ) if

1. $\mu : \Sigma \to (0, \infty]$,
2. $\mu(\emptyset) = 0$,
3. $A_i \in \Sigma$ pairwise disjoint for $i \in \mathbb{N} \Rightarrow \mu\left(\bigcup_{i=0}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

Definition 1.3 (Borel σ-algebra)**.** If (X, τ) is a topological space, then the *Borel σ-algebra* of X, denote by $\mathcal{B}(X)$, is the smallest σ-algebra containing all open sets of X.

Remark 1.4. A σ-algebra is closed under countable intersections and relative complementation.

One goal we would like to achieve is to construct the Lebesgue measure.

Definition 1.5 (Cell in $\mathbb{R}$)**.** A set $I \subseteq \mathbb{R}$ is a *cell* if $(a, b) \subseteq I \subseteq [a, b]$ for some $a, b \in \mathbb{R}$ with $a \leq b$. a is the *left endpoint* of I and b is the *right endpoint* if I.

Definition 1.6 (Cell in $\mathbb{R}^d$)**.** A set $I \subseteq \mathbb{R}^d$ is a *cell* if $I = I_1 \times I_2 \times \cdots \times I_d$ where $I_i \subseteq \mathbb{R}$ is a cell.

Definition 1.7 (Volume of a cell)**.** If $I = I_1 \times I_2 \times \cdots I_d \subseteq \mathbb{R}^d$ is a cell and $a_i \leq b_i$ are the left and right endpoints of each I_i, then the *volume* of I is

$$\ell(I) = \prod_{i=1}^{d} (b_i - a_i).$$

Definition 1.8 (Outer measure)**.** If X is set and Σ is a σ-algebra on X, μ^* is an *outer measure* if

1. $\mu^*(\emptyset) = 0$,
2. (Sub-additivity) $A_i \in \Sigma$ for $i \in \mathbb{N} \Rightarrow \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Definition 1.9 (Lebesgue outer measure)**.** Given $A \subseteq \mathbb{R}^d$, the *Lebesgue outer measure* of A is

$$\lambda^* = \inf\left\{ \sum_{i=1}^{\infty} \ell(I_i) \,\middle|\, \text{for } i \in \mathbb{N},\, I_i \text{ is a cell and } \bigcup_{i=1}^{\infty} I_i \supseteq A \right\}.$$

Remark 1.10. Note that λ^* is defined for all subsets A of $\mathbb{R}^d$, but is only countably additive on a subset of $\mathcal{B}(\mathbb{R}^d)$.

Proposition 1.11. λ^* is an outer measure on $(\mathbb{R}^d, \mathcal{P}(\mathbb{R}^d))$.

Proof. Certainly $\lambda^*(\emptyset) = 0$. Fix $\epsilon > 0$. Let $A_i \subseteq \mathbb{R}^d$ for $i \in \mathbb{N}$. Without loss of generality, assume each $\lambda^*(A_i)$ is finite. For every i, we can find cells $I_{i,j}$ for $j \in \mathbb{N}$ such that $\bigcup_{j=1}^{\infty} I_{i,j} \supseteq I_i$ and $\sum_{j=1}^{\infty} \ell(I_{i,j}) < \lambda^* + \frac{\epsilon}{2^i}$. Certainly, $\{I_{i,j}\}_{i,j\in\mathbb{N}}$ is a cover of $\bigcup_{i=1}^{\infty} A_i$ by cells, so that

$$\lambda^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \ell(I_{i,j}) < \sum_{i=1}^{\infty}\left(\lambda^*(A_i) + \frac{\epsilon}{2^i}\right) \le \epsilon + \sum_{i=1}^{\infty} \lambda^*(A_i).$$

Taking $\epsilon \to 0$ completes the proof. □

Proposition 1.12 (Monotonicity)**.** Outer measures are monotonic.

Proof. Obvious. □

2 August 30, 2013

Proposition 2.1 (Separated additivity)**.** If $A, B \subseteq \mathbb{R}^d$ and $\operatorname{dist}(A, B) > 0$, then $\lambda^*(A) + \lambda^*(B) = \lambda^*(A \cup B)$.

Proof. Without loss of generality, assume that A and B have finite Lebesgue outer measure. By sub-additivity of outer measures, it suffices to show that $\lambda^*(A) + \lambda^*(B) \leq \lambda^*(A \cup B)$. Fix $\epsilon > 0$. We can choose cells I_i for $i \in \mathbb{N}$ such that $\lambda^*(A \cup B) \leq \sum_{i=1}^{\infty} \ell(I_i) \leq \lambda^*(A \cup B) + \epsilon$.

We can subdivide the cells $\mathcal{I} := \{I_i\}_{i \in \mathbb{N}}$ into another set of cells $\mathcal{J}$ that cover $A \cup B$ such that $\operatorname{diam}(J) < \frac{d(A,B)}{2}$ for all $J \in \mathcal{J}$. Choose $\mathcal{K}$ to be all the cells in $\mathcal{J}$ which intersect A. Let $\{J_l''\} = \{J_j\} - \{J_k'\}$. Certianly $J_k' \cap B = \emptyset$, so that $\{J_l''\}$ covers B. Similarly, $\{J_k'\}$ covers A. Hence,

$$\lambda^*(A \cup B) + \epsilon \geq \sum_{i=1}^{\infty} \ell(I_i) = \sum_{j=1}^{\infty} \ell(J_j) = \sum_{k=1}^{\infty} \ell(J_k') + \sum_{l=1}^{\infty} \ell(J_l'') \geq \lambda^*(A) + \lambda^*(B).$$

Sending $\epsilon \to 0$, the result follows. □

We want to eventually show that the Lebesgue outer measure of a cell equals the volume of that cell. To show this, we first prove an easy lemma.

Lemma 2.2. For every $A \subseteq \mathbb{R}^d$,

$$\lambda^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \ell(I_i) \,\middle|\, \text{for } i \in \mathbb{N},\ I_i \text{ is an open cell and } \bigcup_{i=1}^{\infty} I_i \supseteq A \right\}.$$

Proof. Let $\mu^*(A)$ denote the right-hand side of the equality above. Trivially, $\mu^*(A) \geq \lambda^*(A)$. We need to show $\mu^*(A) \leq \lambda^*(A)$.

Pick $\epsilon > 0$. We can find cells $\{I_i\}_{i \in \mathbb{N}}$ that cover A such that $\sum_{i=1}^{\infty} \ell(I_i) \leq \lambda^*(A) + \epsilon$. For each i, we can certainly find an open cell $J_i \supseteq I_i$ such that $\ell(J_i) \leq \ell(I_i) + \frac{\epsilon}{2^i}$. So $\{J_i\}_{i \in \mathbb{N}}$ is an open cover of A, and

$$\mu^*(A) \leq \sum_{i=1}^{\infty} \ell(J_i) \leq \sum_{i=1}^{\infty} \left(\ell(I_i) + \frac{\epsilon}{2^i} \right) = \sum_{i=1}^{\infty} \ell(I_i) + \epsilon \leq \lambda^*(A) + 2\epsilon.$$

Sending $\epsilon \to 0$, we have $\mu^*(A) \leq \lambda^*(A)$. □

Theorem 2.3. For every cell $I \subseteq \mathbb{R}^d$, $\lambda^*(I) = \ell(I)$.

Proof. Trivially, $\lambda^*(I) \leq \ell(I)$, since $\{I\}$ is a cover of I. We need to show $\lambda^*(I) \geq \ell(I)$.

Suppose I is closed. Pick $\epsilon > 0$. By Lemma 2.2, we can find open cells $\{J_i\}_{i \in \mathbb{N}}$ such that $I \subseteq \bigcup_{i \in \mathbb{N}} J_i$ and $\sum_{i=1}^{\infty} \ell(J_i) < \lambda^*(I) + \epsilon$. I is compact, so there is a finite subset of $\{J_i\}$ that covers I. Without loss of generality, assume that the subset is $\{J_1, \ldots, J_N\}$ for some $N \in \mathbb{N}$. So we have $\sum_{i=1}^{N} \ell(J_i) < \lambda^*(I) + \epsilon$.

Extend the faces of each cell J_k to hyperplanes and use these to subdivide I into a finite number of open cells $\{J'_k\}$. Then for each $1 \leq i \leq N$, $\overline{J_k}$ is the union of the closures of some cells in $\{J'_k\}$ and some closed cells outside of I. From this it is easy to see that $\sum_{i=1}^N \ell(J_i) \geq \sum_k \ell(J'_k) = \ell(I)$ where the last equality simply follows from the definition of volume and the distributive laws. So $\ell(I) \leq \sum_{i=1}^N \ell(J_i) \leq \lambda^*(I) + \epsilon$. Sending $\epsilon \to 0$, the result follows.

Suppose I is not closed. We can choose a closed cell $J \subseteq I$ such that $\ell(J) \geq \ell(I) - \epsilon$. From above, we have $\lambda^*(J) = \ell(J) \geq \ell(I) - \epsilon$. So by monotonicity of outer measures, $\lambda^*(I) \geq \lambda^*(J) \geq \ell(I) - \epsilon$. Sending $\epsilon \to 0$, the result follows. $\square$

Next time, we attempt to construct measures from outer measures using the Caratheodory construction.

3 September 4, 2013

Let X be some set.

Today, we will construct measures from outer measures using the Caratheodory construction.

Theorem 3.1 (Caratheodory Construction)**.** Say $\mu^* : \mathcal{P}(X) \to [0, \infty)$ is an outer measure. Define

$$\Sigma = \{E \subseteq X \mid \forall\, A \subseteq X,\, \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)\}\,.$$

Then Σ is a σ-algebra, and $\mu := \mu^*|_\Sigma$ is a measure.

Proof. We prove this through several steps:

1) Certainly, $\emptyset \in \Sigma$.

2) Let $E \in \Sigma$. For any $A \subseteq X$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A \cap (E^c)^c) + \mu^*(A \cap E^c),$$

   so that $E^c \in \Sigma$. So Σ is closed under complementation.

3) Let $E, F \in \Sigma$. Consider any $A \subseteq X$. We have

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap E) + \mu^*(A \cap E^c)\\ &= \mu^*(A \cap (E \cap F)) + \mu^*(A \cap (E \cap F^c)) + \mu^*(A \cap E^c).\end{aligned}$$

   Note $\mu^*(A \cap (E \cap F)^c) = \mu^*(A \cap (E \cap F)^c \cap E) + \mu^*(A \cap (E \cap F)^c \cap E^c)$. We have $E \cap F \subseteq E \Rightarrow (E \cap F)^c \supseteq E^c \Rightarrow (E \cap F)^c \cap E^c = E^c$ and $(E \cap F)^c \cap E = E \cap F^c$, so that $\mu^*(A \cap (E \cap F)^c) = \mu^*(A \cap (E \cap F^c)) + \mu^*(A \cap E^c)$. Therefore, $\mu^*(A) = \mu^*(A \cap (E \cap F)) + \mu^*(A \cap (E \cap F)^c)$, and $E \cap F \in \Sigma$. So Σ is closed under finite intersections.

   As a result, Σ is closed under finite unions.

4) Let $E, F \in \Sigma$ be disjoint. Then for any $A \subseteq X$,

$$\mu^*(E \cup F) = \mu^*((E \cup F) \cap E) + \mu^*((E \cup F) \cap E^c) = \mu^*(E) + \mu^*(F).$$

   So $\mu^*|_\Sigma$ is finitely disjointly additive.

5) Let $\{E_i\} \subseteq \Sigma$. Define $F_n = \bigcup_{i=1}^n E_i$ and $E = \bigcup_{i=1}^\infty E$. Since Σ is closed under finite unions, $F_n \in \Sigma$. Consider any $A \subseteq X$. By sub-additivity, $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$. Also, for each n

$$\begin{aligned}\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c) &\geq \mu^*(A \cap F_n) + \mu^*(A \cap E^c)\\ &= \sum_{i=1}^n \mu^*(A \cap E_i) + \mu^*(A \cap E^c).\end{aligned}$$

Taking $n \to \infty$, it follows that

$$\mu^*(A) \geq \sum_{i=1}^{\infty} \mu^*(A \cap E_i) + \mu^*(A \cap E^c) \geq \mu^* \left(A \cap \bigcup_{i=1}^{\infty} E_i \right) + \mu^*(A \cap E^c)$$
$$= \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

So $E \in \Sigma$, and Σ is closed under countable unions.

From this and the previous points, it follows that Σ is a σ-algebra.

6) Let $\{E_i\} \subseteq \Sigma$ be pairwise disjoint. Define $F_n = \bigcup_{i=1}^n E_i$ and $E = \bigcup_{i=1}^{\infty} E$. $\mu^*|_\Sigma$ is finitely disjointly additive, so $\mu^*(F_n) = \mu^* \left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu^*(E_i)$. Monotonicity implies that $\mu^*(E) \geq \mu^* \left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu^*(E_i)$. Taking $n \to \infty$, this implies that $\mu^*(E) \geq \sum_{i=1}^{\infty} \mu^*(E_i)$. Sub-additivity gives the other direction of the inequality.

   Thus, $\mu^*|_\Sigma$ is a measure.

$\square$

Using the Caratheodory construction, we can now construct the Lebesgue measure from the Lebesgue outer measure.

Definition 3.2 (Lebesgue Measure)**.** Let

$$\mathcal{L} = \left\{ E \subseteq \mathbb{R}^d \,\middle|\, \forall\, A \subseteq \mathbb{R}^d,\, \lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c) \right\}.$$

$\mathcal{L}$ is the *Lebesgue σ-algebra* and $\lambda := \lambda^*|_{\mathcal{L}}$ is the *Lebesgue measure*.

4 September 6, 2013

Typically, sets in σ-algebra Σ are called measurable. In $\mathbb{R}^d$, sets in $\mathcal{L}$ are called Lebesgue measurable and sets in $\mathcal{B}\left(\mathbb{R}^d\right)$ are called Borel measurable.

Definition 4.1 (Null Set)**.** We say $A \subseteq \mathbb{R}^d$ is a *null set* if there exists $E \in \mathcal{L}$ such that $A \subseteq E$ and $\lambda(E) = 0$.

Claim 4.2. If N is a null set, then $N \in \mathcal{L}$ and $\lambda(N) = 0$.

Proof. Consider any $A \subseteq \mathbb{R}^d$. By sub-additivity, $\lambda^*(A) \leq \lambda^*(A \cap N) + \lambda^*(A \cap N^c)$. By monotonicity, $\lambda^*(A \cap N) \leq \lambda^*(N) = 0$ and $\lambda^*(A \cap N^c) \leq \lambda^*(A)$, so that $\lambda^*(A \cap N) + \lambda^*(A \cap N^c) \leq \lambda^*(A)$. $\square$

Claim 4.3. Let $A \subseteq \mathbb{R}^d$. Every $B \subseteq A$ is Lebesgue measurable if and only if $\lambda(A) = 0$.

Proof. This was done in Homework 3. $\square$

Eventually, we will also show that every Lebesgue measurable set is the union of a Borel measurable set and a null set.

Proposition 4.4. $\mathcal{B}\left(\mathbb{R}^d\right) \subseteq \mathcal{L}$.

Proof. Because open sets are countable unions of cells, $\mathcal{B}\left(\mathbb{R}^d\right)$ is generated by open sets, and $\mathcal{L}$ is a σ-algebra, it suffices to show that $I \in \mathcal{L}$ for every cell I. Fix any $A \subseteq \mathbb{R}^d$. By sub-additivity, we have $\lambda^*(A) \leq \lambda^*(A \cap I) + \lambda^*(A \cap I^c)$.

For each n, let $I_n \subseteq I$ be the cell such that $\operatorname{dist}(I_n, I^c) = \frac{1}{n}$. By Proposition 2.1, $\lambda^*(A \cap I^c) + \lambda^*(A \cap I_n) = \lambda^*(A \cap (I^c \cap I_n)) \leq \lambda^*(A)$. Let $B_n = I - I_n$. Then $\lambda^*(A \cap I) \leq \lambda^*(A \cap I_n) + \lambda^*(A \cap B_n) \leq \lambda^*(A \cap I_n) + \lambda^*(B_n)$.

Let M be the maximum side length of I. It is easy to see that $\lambda^*(B_n) \leq \frac{2dM^{d-1}}{n}$. Thus $\lim_{n\to\infty} \lambda^*(B_n) = 0$. Since $\lambda^*(A) \geq \lambda^*(A \cap I^c) + \lambda^*(A \cap I) - \lambda^*(B_n)$, taking $n \to \infty$, it follows that $\lambda^*(A) \geq \lambda^*(A \cap I^c) + \lambda^*(A \cap I)$. $\square$

Later, we will show that there exists sets that are not Lebesgue measurable (given the Axiom of Choice) and sets that are Lebesgue measurable but not Borel measurable.

Proposition 4.5 (Uniqueness of Lebesgue Measure)**.** If μ is any measure on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$ such that $\mu(I) = \lambda(I)$ for every cell I, then $\mu(E) = \lambda(E)$ for every $E \in \mathcal{B}\left(\mathbb{R}^d\right)$.

Proof. Consider any $E \in \mathcal{B}\left(\mathbb{R}^d\right)$. Then for arbitrary cells $\{I_i\}$ that cover E, $\mu(E) \leq \sum_{i=1}^{\infty} \mu(I_i) = \sum_{i=1}^{\infty} \lambda(I_i) = \sum_{i=1}^{\infty} \ell(I_i)$. So $\mu(E) \leq \lambda^*(E) = \lambda(E)$.

Suppose E is bounded. Then we can find a cell $I \supseteq E$. Since $\mu(I) = \lambda(I)$, we have $\mu(I - E) \leq \lambda(I - E) \Rightarrow \mu(E) \geq \lambda(E)$.

If E is unbounded, we have $\lambda(E) = \lim_{n\to\infty} \lambda(E \cap B(0,n))$ (by Homework 1). Since each $(E \cap B(0,n))$ is bounded, we have

$$\mu(E) = \lim_{n\to\infty} \mu(E \cap B(0,n)) = \lim_{n\to\infty} \lambda(E \cap B(0,n)) = \lambda(E).$$

$\square$

5 September 9, 2013

Say Σ is a σ-algebra of X and $\mathcal{C} \subseteq \Sigma$. The goal today is to determine what properties $\mathcal{C}$ should have such that if two measures μ and ν agree on $\mathcal{C}$, then they agree on $\sigma(C)$. For a general $\mathcal{C}$, it is not the case that the two measures should agree on $\mathcal{C}$.

Example 5.1. If $A, B \in \Sigma$ and $\mu(A) = \nu(A)$ and $\mu(B) = \nu(B)$, it need not be that $\mu(A \cap B) = \nu(A \cap B)$. For example, take $X = \{1, 2, 3\}$, $A = \{1, 2\}$, $B = \{2, 3\}$, and let $\mu(\{1\}) = \mu(\{3\}) = 0$, $\mu(\{2\}) = 1$, $\nu(\{1\}) = \nu(\{3\}) = 0$, and $\nu(\{2\}) = 0$. So one might want $\mathcal{C}$ to be closed under finite intersections.

Example 5.2. If $A \in \Sigma$ and $\mu(A) = \nu(A)$, it need not be that $\mu(A^c) = \nu(A^c)$ if $\mu(A), \mu(X) = \infty$. So one might want μ and ν to be finite measures.

Definition 5.3 (π-system)**.** We say $\mathcal{C} \subseteq \mathcal{P}(X)$ is a π-*system* if $\mathcal{C}$ is closed under finite intersections.

Definition 5.4 (λ-system)**.** We say $\Lambda \subseteq \mathcal{P}(X)$ is a λ-*system* if

1. $X \in \Lambda$;
2. if $A_i \in \Lambda$ with $A_i \subseteq A_{i+1}$ for $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} A_i \in \Lambda$;
3. if $A, B \in \Lambda$ with $A \subseteq B$, then $B - A \subseteq \Lambda$.

Definition 5.5. The intersection of λ-systems is a λ-system. Therefore, we can define $\lambda(\mathcal{C}) := \bigcap_{\Lambda \in S} \Lambda$ to be the smallest λ-system containing $\mathcal{C}$, where $S = \{\Lambda \mid \Lambda \supseteq \mathcal{C} \text{ is a } \lambda\text{-system}\}$.

Remark 5.6. If Λ is a λ-system and a π-system, then Λ is a σ-algebra.

Proof. $X \in \Lambda$, so for any $A \in \Lambda$, $A^c = X - A \in \Lambda$. In particular, $X - X = \emptyset \in \Lambda$.

If $A, B \in \Lambda$ are disjoint, then $A \subseteq B^c \Rightarrow B^c - A = (A \cup B)^c \in \Lambda$. Since $X \in \Lambda$, $X - (A \cup B)^c = A \cup B \in \Lambda$. So Λ is closed under finite disjoint unions. So for any $A, B \in \Lambda$, $A \cap B \in \Lambda$, so that $A - (A \cap B), B - (B \cap B) \in \Lambda$. This implies $A \cup B = (A \cap B) \cup (A - (A \cap B)) \cup (B - (A \cap B)) \in \Lambda$. So Λ is closed under finite unions.

Consider any $A_i \in \Lambda$ for $i \in \mathbb{N}$. Define $B_n = \bigcup_{i=1}^{n} A_i$ for each $n \in \mathbb{N}$. Then $B_n \subseteq B_{n+1}$, and $B_n \in \Lambda$ by what we just showed above. So $\bigcup_{n=1}^{\infty} B_n \in \Lambda$. Since $\bigcup_{n=1}^{\infty} B_n = \bigcup_{i=1}^{\infty} A_i$, it follows that $\bigcup_{i=1}^{\infty} A_i \in \Lambda$, and Λ is closed under countable unions. $\square$

The following theorem shows that the added assumptions of $\mathcal{C}$ suggested by the above examples suffices.

Theorem 5.7. If $\mathcal{C}$ is a π-system and $\Lambda \supseteq \mathcal{C}$ is a λ-system, then $\Lambda \supseteq \sigma(\mathcal{C})$. In particular, $\lambda(\mathcal{C}) = \sigma(\mathcal{C})$.

Proof. It will suffice to show that $\lambda(\mathcal{C})$ is a π-system (since by Remark 5.6, this implies that $\lambda(\mathcal{C})$ is a σ-algebra containing $\mathcal{C}$, implying that $\Lambda \supseteq \lambda(\mathcal{C}) \supseteq \sigma(\mathcal{C})$).

Fix $A \in \mathcal{C}$, and let $\Lambda' := \{B \in \lambda(\mathcal{C}) \mid A \cap B \in \lambda(\mathcal{C})\}$. Certainly $X \in \Lambda'$.

Suppose $E, F \in \Lambda'$ with $E \subseteq F$. Then $A \cap E$ and $A \cap F \in \lambda(\mathcal{C})$ with $A \cap E \subseteq A \cap F$. So $(A \cap F) - (A \cap E) = A \cap (F - E) \in \lambda(\mathcal{C})$ and $F - E \in \Lambda'$.

Suppose $E_i \in \Lambda'$ with $E_i \subseteq E_{i+1}$ for each $i \in \mathbb{N}$. Then $A \cap E_i \in \lambda(\mathcal{C})$ with $A \cap E_i \subseteq A \cap E_{i+1}$ for each $i \in \mathbb{N}$. So $\bigcup_{i=1}^{\infty}(A \cap E_i) = A \cap \bigcup_{i=1}^{\infty} E_i \in \lambda(\mathcal{C})$, so that $\bigcup_{i=1}^{\infty} E_i \in \Lambda'$.

So Λ' is a λ-system. In particular, by definition of $\lambda(\mathcal{C})$, $\Lambda' = \lambda(\mathcal{C})$.

Now, fix $A \in \lambda(\mathcal{C})$, and let $\Lambda'' := \{B \in \lambda(\mathcal{C}) \mid A \cap B \in \lambda(\mathcal{C})\}$. The exact same argument as above shows that $\Lambda'' = \lambda(\mathcal{C})$ is a λ-system. In particular, we showed that $\lambda(\mathcal{C})$ is closed under finite intersections, so that $\lambda(\mathcal{C})$ is a π-system.

By definition of σ-algebras, $\sigma(\mathcal{C}) \supseteq \lambda(\mathcal{C})$. By choosing $\Lambda = \lambda(\mathcal{C})$, we have that $\lambda(\mathcal{C}) \supseteq \sigma(\mathcal{C})$. Thus, $\lambda(\mathcal{C}) = \sigma(\mathcal{C})$. $\square$

Corollary 5.8. If μ and ν are two finite measures that agree on a π-system $\mathcal{C}$, then $\mu = \nu$ on $\sigma(C)$.

Proof. Let $\Lambda = \{E \subseteq \lambda(C) \mid \mu(E) = \nu(E)\}$. Since measures are countably additive, disjointly additive, and monotonic, it easily follows that $\Lambda = \lambda(C)$. By Theorem 5.7, it follows that μ and ν agree on $\lambda(C) = \sigma(C)$. $\square$

We can use this to come up with a cleaner proof that the Lebesgue measure is unique.

Corollary 5.9. If μ is any measure on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$ such that $\mu(I) = \lambda(I)$ for every cell I, then $\mu(E) = \lambda(E)$ for every $E \in \mathcal{B}\left(\mathbb{R}^d\right)$.

Proof. Let $\mathcal{C}$ be the set of all cells in $\mathbb{R}^d$. Certainly $\mathcal{C}$ is a π-system. For $A \in \mathcal{L}$, define $\mu_n(A) := \mu(A \cap B(0,n))$ and $\lambda_n(A) := \lambda(A \cap B(0,n))$. Certainly μ_n and λ_n are finite measures that agree on $\mathcal{C}$. Hence, by Corollary 5.8, $\mu_n = \lambda_n$ on $\sigma(\mathcal{C}) \supseteq \mathcal{B}\left(\mathbb{R}^d\right)$. Further, note that for $A \in \mathcal{L}$, $\mu(A) = \lim_{n\to\infty} \mu_n(A)$ and $\lambda(A) = \lim_{n\to\infty} \lambda_n(A)$ (this actually follows from Homework 1). So, it follows that $\mu = \lambda$ on $\mathcal{B}\left(\mathbb{R}^d\right)$. $\square$

6 September 11, 2013

The goal today is to describe measures that can be approximated arbitrarily well by nice sets.

Definition 6.1 (Regular Measure)**.** Let X be a metric space and $\mathcal{B}(X)$ the Borel σ-algebra of X. We say μ is a *regular Borel measure* in X if

1. μ is a measure on $(X, \mathcal{B}(X))$;
2. for every $A \in \mathcal{B}(X)$, $\mu(A) = \inf\{\mu(U) \,|\, U \supseteq A \text{ is open}\}$;
3. for every $A \in \mathcal{B}(X)$, $\mu(A) = \sup\{\mu(K) \,|\, K \subseteq A \text{ is compact}\}$;
4. for every compact $K \in \mathcal{B}(X)$, $\mu(K) < \infty$.

A measure satisfying condition 2 is called an *outer regular measure.* A measure satisfying condition 3 is called an *inner regular measure.* A measure satisfying condition 4 is called a *Radon measure.*

If $A \in \mathcal{B}(X)$ satisfies condition 2, then call A *inner regular with respect to* μ, or μ*-inner regular.* If A satisfies condition 3, then call A *outer regular with respect to* μ, or μ*-outer regular.*

Remark 6.2. Sometimes, in locally compact Hausdorff spaces, only open sets need to be μ-inner regular.

Our goal is to prove that the Lebesgue measure is regular.

Theorem 6.3 (Regularity of Finite Borel Measures)**.** Let X be a compact space. Let μ be any finite Borel measure on X. Then μ is regular.

Proof. Since μ is finite, it is automatically Radon.

Let $\Lambda = \{A \in \mathcal{B}(X) \,|\, A \text{ is inner and outer regular with respect to } \mu\}$. It suffices to show that Λ contains all open sets and that Λ is a λ-system (since the set of all open sets is a π-system, implying by Theorem 5.7 that $\Lambda \supseteq \mathcal{B}(X)$).

Let $U \subseteq X$ be open. Trivially, U is μ-outer regular. Define

$$K_n = \left\{ x \in U \,\middle|\, \operatorname{dist}(x, U^c) \geq \frac{1}{n} \right\}.$$

Certainly, K_n is closed. Since X is compact, K_n is compact. Since U is open, $x \in U$ if and only if $d(x, U) > 0$. Therefore, $U = \bigcup_{n=1}^{\infty} K_n$. Since $K_n \subseteq K_{n+1}$, $\lim_{n\to\infty} \mu(K_n) \to \mu(U)$. So U is μ-inner regular. Thus, Λ contains all open sets.

Since X is the whole space, X is open and compact. So $X \in \Lambda$.

Consider $A_1, A_2 \in \Lambda$ with $A_1 \subseteq A_2$. Fix $\epsilon > 0$. For each i, we can find open U_i and compact K_i such that $K_i \subseteq A_i \subseteq U_i$ and $\mu(A_i - K_i) < \epsilon$ and $\mu(U_i - A_i) < \epsilon$. Note that $K_1 \cap U^c \subseteq A_1 - A_2 \subseteq U_1 - K_2$ and that $K_1 \cap U^c$ is compact and

$U_1 - K - 2$ is open. Further,

$$\begin{aligned}\mu((U_1 - K_2) - (A_1 - A_2)) &= \mu((U_1 - A_1) \cup (A_2 - K_2)) \\ &= \mu(U_1 - A_1) + \mu(A_2 - K_2) < 2\epsilon\end{aligned}$$

and

$$\begin{aligned}\mu((A_1 - A_2) - (K_1 \cap U_2^c)) &= \mu((A_1 \cap A_2^c) \cap (K_1 \cap U_2^c)^c) \\ &= \mu((A_1 \cap A_2^c) \cap (K_1^c \cup U_2)) \\ &= \mu((A_1 \cap A_2^c \cap K_1^c) \cup (A_1 \cap A_2^c \cap U_2)) \\ &\leq \mu((A_1 \cap K_1^c) \cup (U_2 \cap A_2^c)) \\ &\leq \mu(A_1 \cap K_1^c) + \mu(U_2 \cap A_2^c) < 2\epsilon.\end{aligned}$$

Taking $\epsilon \to 0$, it follows that $A_1 - A_2 \in \Lambda$.

Consider $A_i \in \Lambda$ with $A_i \subseteq A_{i+1}$. Fix $\epsilon > 0$. For each i, we can find open U_i and compact K_i such that $K_i \subseteq A_i \subseteq U_i$ and $\mu(A_i - K_i) < \frac{\epsilon}{2^i}$ and $\mu(U_i - A_i) < \frac{\epsilon}{2^i}$. Let $A = \bigcup_{i=1}^\infty A_i$ and $U = \bigcup_{i=1}^\infty U_i$. Certainly $A \subseteq U$ with U open, and

$$\mu(U - A) = \mu\left(\bigcup_{i=1}^\infty (U_i - A)\right) \leq \sum_{i=1}^\infty \mu(U_i - A) \leq \sum_{i=1}^\infty \mu(U_i - A_i) < \epsilon.$$

Let $E_n = \bigcup_{i=1}^n K_i$ and $K = \bigcup_{i=1}^\infty K_i$. Note each E_n is compact, and $E_n \subseteq E_{n+1} \subseteq K$. Then $\lim_{n\to\infty} \mu(E_n) = \mu(K) < \infty$, so that we can find N such that $\mu(E_N) > \mu(K) - \epsilon$, or $\mu(K - E_N) < \epsilon$. Then $E_N \subseteq A$, and

$$\begin{aligned}\mu(A - E_N) = \mu(A - K) + \mu(K - E_N) &= \mu\left(\bigcup_{i=1}^\infty (A_i - K)\right) + \epsilon \\ &\leq \sum_{i=1}^\infty \mu(A_i - K) + \epsilon \leq \sum_{i=1}^\infty \mu(A_i - K_i) + \epsilon < 2\epsilon.\end{aligned}$$

Taking $\epsilon \to 0$, it follows that $A \in \Lambda$.

Thus, Λ is a λ-system. $\square$

This proof can easily be extended to a more general space.

Theorem 6.4 (Regularity of Borel Measure)**.** If $X = \bigcup_{n=1}^\infty B_n$ where $\overline{B_n}$ is compact, $\overline{B_n} \subseteq B_{n+1}^\circ$, and $\mu(B_n) < \infty$, then μ is regular.

Proof. This was on Homework 3, so the proof is omitted. $\square$

Corollary 6.5 (Lebesgue Measure is Regular)**.** λ is regular.

Proof. Certainly $\mathbb{R}^d = \bigcup_{n=1}^\infty B(0,n)$. Certainly $\overline{B(0,n)} \subseteq B(0,n+1)^\circ = B(0,n+1)$ and $\lambda(B(0,n)) < \infty$ for each n. By Theorem 6.4, it follows that λ is regular. $\square$

7 September 13, 2013

Today, the goal is to construct a non-measurable set, and that all Lebesgue measurable sets is the union of a Borel measurable set and a null set.

Theorem 7.1 (Existence of Non-measurable Sets)**.** There exists $A \subseteq \mathbb{R}$ that is not Lebesgue-measurable.

Proof. Let $\mathcal{C}$ be the set of cosets of $\frac{\mathbb{R}}{\mathbb{Q}}$, where we are considering $(\mathbb{R}, +)$ as a group and $\mathbb{Q}$ as a subgroup of $\mathbb{R}$. Let $A \subseteq \mathbb{R}$ be some set such that A contains exactly one representative of each coset in $\mathcal{C}$. Suppose A is measurable.

Suppose A was Lebesgue measurable. Note that $\{A + q \mid q \in \mathbb{Q}\}$ are disjoint, and that $\mathbb{R} = \bigcup_{q\in\mathbb{Q}}(A + q)$. If $\lambda(A) = 0$, then

$$\lambda(\mathbb{R}) = \lambda\left(\bigcup_{q\in\mathbb{Q}}(A+q)\right) \leq \sum_{q\in\mathbb{Q}} \lambda(A+q) = 0,$$

a contradiction. Thus, $\lambda(A) > 0$.

Now, consider any compact $K \subseteq A$. Let $C = \bigcup_{q\in\mathbb{Q},\,|q|<1}(K+q)$. Certainly, C is bounded, so that it has finite measure. Since $\lambda(K) = \lambda(K+q)$ for every $q \in \mathbb{Q}$, it must be that $\lambda(K) = 0$. This contradicts the regularity of λ, so A must not have been Lebesgue measurable. □

Theorem 7.2. There exists $A \subseteq \mathbb{R}$ such that

1. if $E \subseteq A$ is Lebesgue measurable, then $\lambda(E) = 0$;
2. if $E \subseteq A^c$ is Lebesgue measurable, then $\lambda(E) = 0$.

Proof. TODO □

Lemma 7.3. For every $A \in \mathcal{L}(\mathbb{R}^d)$, for every $\epsilon > 0$, there exists open U and closed C such that $C \subseteq A \subseteq U$ and $\lambda(U - C) < \epsilon$.

Proof. Consider any $A \in \mathcal{L}(\mathbb{R}^d)$. Fix $\epsilon > 0$. Let $A_n = A \cap (B(0,n) - B(0,n-1))$, where $B(0,0) = \emptyset$. Then $A = \bigcup_{n=1}^{\infty} A_n$. Further, note $\lambda(A_n) < \infty$ for each n, so that by regularity of λ, we can find compact K_n and open U_n such that $\lambda(A_n - K_n) < \frac{\epsilon}{2^{n+1}}$ and $\lambda(U_n - A_n) < \frac{\epsilon}{2^{n+1}}$. Denote $C = \bigcup_{n=1}^{\infty} K_n$ and $U = \bigcup_{n=1}^{\infty} U_n$. Certainly $C \subseteq A \subseteq U$. Further, since the A_n's are disjoint, the K_n's are disjoint, so that C must be closed (since the union of disjoint closed sets is closed). Then

$$\lambda(U - A) = \lambda\left(\bigcup_{n=1}^{\infty}(U_i - A)\right) \leq \sum_{n=1}^{\infty} \lambda(U_i - A) \leq \sum_{n=1}^{\infty} \lambda(U_i - A_i) < \frac{\epsilon}{2}.$$

Similarly, $\lambda(A - C) < \frac{\epsilon}{2}$. So, $\lambda(U - C) < \epsilon$. □

8 September 16, 2013

Theorem 8.1. For every $A \in \mathcal{L}(\mathbb{R}^d)$, there exists $B \in \mathcal{B}\left(\mathbb{R}^d\right)$ and a null set N such that $A = B \cup N$.

Proof. Let $A \in \mathcal{L}(\mathbb{R}^d)$. By Lemma 7.3, for $n \in \mathbb{N}$, we can find open U_n and closed C_n such that $C_n \subseteq A \subseteq U_n$ and $\lambda(U_n - C_n) < \frac{1}{n}$. Choose $B = \bigcup_{n=1}^{\infty} C_n$. Then

$$\lambda(A - B) \leq \lambda\left(U_n - \bigcup_{n=1}^{\infty} C_n\right) \leq \lambda(U_n - C_n) < \frac{1}{n}.$$

Taking $n \to \infty$, it follows that $\lambda(A - B) = 0$. So choosing $N = A - B$ completes the proof. $\square$

Let (X, Σ, μ) be a measure space.

Definition 8.2. Define $\mathcal{N} = \{A \subseteq X \mid \exists\, B \in \Sigma,\, B \supseteq A,\, \mu(B) = 0\}$ to be the set of all μ-null sets.

Definition 8.3 (Complete Measure Space)**.** The σ-algebra Σ is *complete* with respect to μ, or μ-complete, if $\Sigma \supseteq \mathcal{N}$.

Remark 8.4. The σ-algebra of a measure constructed via an outer measure using the Caratheodory construction is complete.

Proof. Let μ be a measure on X constructed via an outer measure using the Caratheodory construction. Let N be any μ-null set. For any $A \in X$, we can find $M \in \Sigma$ such that $\mu(M) = 0$ and $M \supseteq N$. Then $\mu^*(N) \leq \mu^*(M) = 0$, so that $\mu^*(N) = 0$. Therefore,

$$\mu^*(A \cap N) + \mu^*(A \cap N^c) \leq \mu^*(N) + \mu^*(A \cap N^c) \leq 0 + \mu^*(A).$$

By sub-additivity, $\mu^*(A \cap N) + \mu^*(A \cap N^c) \geq \mu^*(A)$. Thus, it follows that $N \in \Sigma$. So Σ is complete. $\square$

Corollary 8.5. $\mathcal{L}(\mathbb{R}^d)$ is complete.

Definition 8.6 (Completion of Measure Space)**.** Σ_μ is the *completion* of Σ with respect to μ if

$$\Sigma_\mu = \{A \cup N \mid A \in \Sigma,\, N \in \mathcal{N}\}.$$

Definition 8.7. For every $A \in \Sigma_\mu$, define a measure $\overline{\mu}(A) = \mu(B)$ if $A = B \cup N$ where $B \in \Sigma$ and $N \in \mathcal{N}$.

Claim 8.8. $A \in \Sigma_\mu$ if and only if there are $F, G \in \Sigma$ such that $F \subseteq A \subseteq G$ and $\mu(G - F) = 0$.

Proof. If $A \in \Sigma_\mu$, there are $B \in \Sigma$ and $N \in \mathcal{N}$ such that $A = B \cup N$. Since N is μ-null, there is $M \in \Sigma$ such that $M \supseteq N$ and $\mu(M) = 0$. Choose $F = B$ and $G = B \cap M$. Then $F \subseteq A \subseteq G$, and $\mu(G - F) = \mu((B \cup M) - B) \leq \mu(M) = 0$.

Conversely, suppose there are $F, G \in \Sigma$ such that $F \subseteq A \subseteq G$ and $\mu(G - F) = 0$. Then choose $B = F$ and $N = A - F$. Clearly $B \in \Sigma$ and $N \in \mathcal{N}$ and $A = B \cup N$. $\square$

Proposition 8.9. Σ_μ is a σ-algebra, $\overline{\mu}$ is a measure on Σ_μ, $\overline{\mu}|_\Sigma = \mu$, and Σ_μ is complete with respect to μ.

Proof. Certainly $\emptyset \in \Sigma_\mu$. If $A \in \Sigma_\mu$, then by Claim 8.8, we can find $F, G \in \Sigma$ such that $F \subseteq A \subseteq G$ and $\mu(F - G) = 0$. Clearly, $G^c \subseteq A^c \subseteq F^c$, and $\mu(F^c - G^c) = \mu(F^c \cap G) = \mu(G - F) = 0$. So by the same claim, $A^c \in \Sigma_\mu$. If $A_i \in \Sigma_\mu$ for $i \in \mathbb{N}$, we can find $B_i \in \Sigma$ and $N_i \in \mathcal{N}$ such that $A_i = B_i \cup N_i$. Let $N = \bigcup_{i=1}^\infty N_i$ and $B = \bigcup_{i=1}^\infty B_i$. Let $A = \bigcup_{i=1}^\infty A_i$. Then $A = B \cup N$. Further, for each N_i, we can find $M_i \subseteq N_i$ with $M_i \in \Sigma$ and $\mu(M_i) = 0$. Let $M = \bigcup_{i=1}^\infty$. Then $\mu(M) = 0$, and $M \supseteq N$, so that $N \in \mathcal{N}$. Thus, $A \in \Sigma_\mu$. Thus, Σ_μ is a σ-algebra.

Choose any $A \in \Sigma_\mu$. Suppose $A = B_1 \cup N_1 = B_2 \cup N_2$ where $B_1, B_2 \in \Sigma$ and $N_1, N_2 \in \mathcal{N}$. We then can find $M_i \supseteq N_i$ with $M_i \in \Sigma$ and $\mu(M_i) = 0$ for each i. Then $\mu(B_1) \leq \mu(B_2 \cup M_2) \leq \mu(B_2) + \mu(M_2) = \mu(B_2)$. Similarly, $\mu(B_2) \leq \mu(B_1)$, so that $\mu(B_1) = \mu(B_2)$. Thus, $\overline{\mu}$ is well-defined, and $\overline{\mu}|_\Sigma = \mu$.

Let $N \subseteq X$ such that there is an $M \in \Sigma_\mu$ with $M \supseteq N$ and $\overline{\mu}(M) = 0$. Then we can find $B \in \Sigma$ and $S \in \mathcal{N}$ such that $M = B \cup S$. Further, $\mu(B) = 0$. So without loss of generality, we may assume $B = \emptyset$, so that $N \subseteq S$. So $N \in \mathcal{N}$. Then we can write $N = \emptyset \cap N$, so that $\mathcal{N} \in \Sigma_\mu$. $\square$

9 September 18, 2013

Definition 9.1 (Measurable Functions)**.** Let X be a set and Σ a σ-algebra on X. Let (Y, τ) be a topological space. We say $f : X \to Y$ is *measurable with respect to* Σ, or Σ-*measurable* if for every open $U \subseteq Y$, $f^{-1}(U) \in \Sigma$.

Remark 9.2. Most of the time, $Y = \mathbb{R}$ and τ is the set of open sets.

Example 9.3. If (X, d) is a metric space and $f : X \to Y$ is continuous, then f is Borel measurable (in other words, f is measurable with respect to $\mathcal{B}(X)$).

Lemma 9.4. If $f : X \to Y$ is any function, let $\mathcal{C} = \left\{B \subseteq Y \,\middle|\, f^{-1}(B) \in \Sigma\right\}$. Then $\mathcal{C}$ is a σ-algebra.

Proof. Certainly $\emptyset \in \mathcal{C}$. Suppose $A \in \mathcal{C}$, so that $f^{-1}(A) \in \Sigma$. Then $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c \in \Sigma$, so that $A^c \in \mathcal{C}$. Suppose $A_i \in \mathcal{C}$ for $i \in \mathbb{N}$, so that $f^{-1}(A_i) \in \Sigma$ for each i. Then $f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i) \in \Sigma$, so that $\bigcup_{i=1}^{\infty} A_i \in \Sigma$. So $\mathcal{C}$ is a σ-algebra. $\square$

Proposition 9.5. Say $f : X \to Y$ is Σ-measurable. For every $B \in \mathcal{B}(Y)$, $f^{-1}(B) \in \Sigma$.

Proof. Let $\mathcal{C} := \left\{B \subseteq Y \,\middle|\, f^{-1}(B) \in \Sigma\right\}$. By Lemma 9.4, $\mathcal{C}$ is a σ-algebra. By definition of measurable functions, $\mathcal{C}$ contains all open sets. Thus, $\mathcal{C} \supseteq \mathcal{B}(Y)$. $\square$

Corollary 9.6. Say $f : X \to \mathbb{R}$. Then f is Σ-measurable if any one of the following holds:

1. for every $\alpha \in \mathbb{R}$, $\{x \in X \mid f(x) < \alpha\} \in \Sigma$;
2. for every $\alpha \in \mathbb{R}$, $\{x \in X \mid f(x) > \alpha\} \in \Sigma$;
3. for every $\beta \in \mathbb{R}$, $\{x \in X \mid f(x) \leq \beta\} \in \Sigma$;
4. for every $\beta \in \mathbb{R}$, $\{x \in X \mid f(x) \geq \beta\} \in \Sigma$.

Proof. Let $\mathcal{C} := \left\{B \subseteq Y \,\middle|\, f^{-1}(B) \in \Sigma\right\}$. We know C contains all intervals $(-\infty, \alpha)$ for $\alpha \in \mathbb{R}$. By Lemma 9.4, $\mathcal{C}$ is a σ-algebra, and so is closed under complementation. So $[\beta, \infty) \in \mathcal{C}$ for all $\beta \in \mathbb{R}$, so that $[\beta, \alpha) \in \mathcal{C}$ for all $\beta < \alpha$. Open sets can be written as a countable union of such intervals, so that $\mathcal{C}$ contains all open sets. So $\mathcal{C} \supseteq \mathcal{B}(Y)$.

The proofs for the remaining three parts are similar. $\square$

Corollary 9.7. Say $f : X \to Y$ is measurable and $g : Y \to Z$ is Borel measurable. Then $g \circ f : X \to Z$ is measurable.

Proof. Consider any open set $U \subseteq Z$. Then $g^{-1}(U) \in \mathcal{B}(Y)$ since g is Borel measurable. Since f is measurable, by Proposition 9.5, $f^{-1}(g^{-1}(U)) \in \Sigma$. Thus, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \in \Sigma$, so that $g \circ f$ is Σ-measurable. $\square$

Proposition 9.8. If $f : X \to \mathbb{R}^m$ and $g : X \to \mathbb{R}^n$ are measurable, then $(f, g) : X \to \mathbb{R}^m \times \mathbb{R}^n$ is measurable.

Proof. Say $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are open. Then $(f,g)^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V) \in \Sigma$. Let $\mathcal{C} = \left\{ U \subseteq \mathbb{R}^{m+n} \,\middle|\, (f,g)^{-1}(U) \in \Sigma \right\}$. By Lemma 9.4, $\mathcal{C}$ is a σ-algebra. Further, $\mathcal{C}$ contains all open cells, so that $\mathcal{C} \supseteq \mathcal{B}(\mathbb{R}^{m+n})$. □

Corollary 9.9. If $f, g : X \to \mathbb{R}$ are measurable, then $f+g, f-g, fg : X \to \mathbb{R}$ is measurable. If g is non-zero, then $\frac{f}{g} : X \to \mathbb{R}$ is measurable.

Proof. Define $h : X \to R \times R$ and $j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $h(x) = (f(x), g(x))$ and $j(x,y) = x + y$. By Proposition 9.8, h is measurable. j is continuous so that it is Borel measurable. So by Corollary 9.7, $f + g = j \circ h$ is measurable. The same argument works for the other three functions. □

Proposition 9.10. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a countable set of measurable functions from X to Y. Then

1. $\inf_{n\in\mathbb{N}} f_n$ is measurable;
2. $\sup_{n\in\mathbb{N}} f_n$ is measurable;
3. $\liminf_{n\to\infty} f_n$ is measurable;
4. $\limsup_{n\to\infty} f_n$ is measurable;
5. if $f_n \to f$ pointwise, then f is measurable.

Proof. Let $g(x) = \inf_{n\in\mathbb{N}} f_n(x)$. Note that

$$\{x \in X \mid g(x) \geq \alpha\} = \bigcup_{n\in\mathbb{N}} \{x \in X \mid f_n(x) \geq \alpha\}.$$

Each $\{x \in X \mid f_n(x) \geq \alpha\} = f_n^{-1}((-\infty, \alpha])$ is certainly measurable, so that the union on the right is certainly measurable. So by Corollary 9.6, it follows that g is measurable as well.

Let $f_n' = -f_n$. Clearly each f_n' is measurable. Then $\sup_{n\in\mathbb{N}} f_n = -\inf_{n\in\mathbb{N}} f_n'$. Since $\inf_{n\in\mathbb{N}} f_n'$ is measurable, it follows that $\sup_{n\in\mathbb{N}} f_n$ is measurable.

$\liminf_{n\to\infty} f_n = \sup_{n>0} \inf_{m\geq n} f_m$. From what we just showed above, it clearly follows that $\liminf_{n\to\infty} f_n$ is measurable. Similarly, $\limsup_{n\to\infty} f_n$ is measurable. Thus, it $\lim_{n\to\infty} f_n$ exists, then it is measurable as well. □

10 September 20, 2013

Today, our goal is to construct a Lebesgue measurable set that is not Borel measurable.

Definition 10.1 (Devil's Staircase)**.** Let $C \subseteq [0,1]$ be the Cantor set. Let α be the Hausdorff dimension of C ($\alpha = \log_3 2$). Let H_α denote the Hausdorff measure of dimension α. Let $F : [0,1] \to [0,1]$ be defined by

$$F(x) = \frac{H_\alpha(C \cap [0,x])}{H_\alpha(C)}.$$

This function F is called the *Devil's Staircase*.

Remark 10.2. The Devil's Staircase F is increasing, continuous, and differentiable almost where with derivative 0.

Theorem 10.3. $\mathcal{B} \subsetneq \mathcal{L}$

Proof. Let F be the Devil's Staircase, and define $g(x) = \inf f^{-1}(\{x\})$. Then note that $f(g(x)) = x$, since f is continuous, and observe that $g([0,1]) = C$, where C is the Cantor set. Pick $B \subseteq [0,1]$ such that $B \notin \mathcal{L}$. We can pick such a set due to Theorem 7.1. Then note that $g(B) \subseteq C \in \mathcal{N}$, so that $g(B) \in \mathcal{L}$.

Now, certainly g is measurable (since g is increasing), so that $g^{-1}(E) \in \mathcal{L}$ for every $E \subseteq \mathcal{B}(\mathbb{R})$. However, note that $g^{-1}(g(B)) = B$. Since $B \notin \mathcal{L}$, $g(B) \notin \mathcal{B}(\mathbb{R})$. So $g(B) \in \mathcal{L} - \mathcal{B}(\mathbb{R})$. $\square$

Definition 10.4 (Almost Everywhere)**.** Let (X, Σ, μ) be a measure space. We say a property P holds *almost everywhere with respect to* μ, or μ*-a.e.*, if there is a μ-null set N and P holds on N^c. If μ is implicitly known, then we simply say almost everywhere, or a.e.

Example 10.5. If f is Riemann integrable, then f is continuous almost everywhere.

Proposition 10.6. Say (X, Σ, μ) is a complete measure space, and let (Y, τ) be a topological space. Let $f, g : X \to \tau$ be functions such that $f = g$ μ-a.e. Then if f is measurable, then g is measurable.

Proof. We can find a μ-null set N such that $f = g$ on N^c. Let U be open. Then $g^{-1}(U) = N^c \cap g^{-1}(U) + N \cap g^{-1}(U) = N^c \cap f^{-1}(U) + N \cap g^{-1}(U)$. Since Σ is complete, the first term in the sum is certainly measurable, and the second term is μ-null, so that $g^{-1}(U) \in \Sigma$. Hence, it follows that g is measurable. $\square$

11 September 23, 2013

Definition 11.1 (Simple Function)**.** Let (X, Σ, μ) be a measure space. A function $s : X \to \mathbb{R}$ is *simple* if s is measurable and has finite range lying in $\mathbb{R}$.

Definition 11.2 (Characteristic Function)**.** Let $A \subseteq X$. Define $\chi_A : X \to \mathbb{R}$ by

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

This function is called the *characteristic function* of A.

Note 11.3. If A is measurable, then χ_A is simple. Suppose s is simple. Then the range of s is $\{a_1, \ldots, a_n\}$ for some $a_i \in \mathbb{R}$ and $n \in \mathbb{N}$. Let $A_i = s^{-1}(\{a_i\})$. Then $s = \sum_{i=1}^n a_i \chi_{A_i}$.

Proposition 11.4. Say $f : X \to [0, \infty]$ is measurable. Then there exists a sequence of simple functions $\{s_n\}_{n \in \mathbb{N}}$ such that $s_n \nearrow f$ pointwise.

Proof. Suppose f was bounded. Without loss of generality, say $f : X \to [0, 1)$. Then for $n, k \in \mathbb{N}$ with $0 \le k < 2^n$, define $A_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, n\frac{k+1}{2^n}\right)\right)$. Define $s_n = \sum_{k=0}^{2^n-1} \frac{k}{2^n} \chi_{A_{n,k}}$. Since f is measurable, $A_{n,k} \in \Sigma$, so s_n is simple. Note that $0 \le f - s_n < \frac{1}{2^n}$, and that $f \ge s_{n+1} \ge s_n$ (since $A_{n,k} = A_{n,2k} \cup A_{n,2k+1}$). So $s_n \nearrow f$ pointwise (in particular, uniformly).

Suppose f was unbounded. Let $A_n = f^{-1}([n, n+1))$, and define $f_n = f\chi_{A_n}$. Since f is measurable, A_n is measurable, so that f_n is measurable. Furthermore, f_n is bounded, so by repeating the above process, we can find simple functions $\{s_{n,m}\}_{m \in \mathbb{N}}$ such that $s_{n,m} \le s_{n,m+1} \le f_n$ and $|f_n - s_{n,m}| < \frac{1}{2^m}$. Note that since $f_n = 0$ on A_n^c, each $s_{n,m} = 0$ on A_n^c.

Now, define $t_n = \sum_{i=1}^n s_{i,n}$. Clearly, $t_n \le t_{n+1}$, and since the A_n's are disjoint, $t_n \le \sum_{i=1}^n f\chi_{A_n} \le f$. Consider any $x \in X$. Then $x \in A_N$ for some N. So for $n \ge N$, $f(x) - t_n(x) = f_N(x) - s_{N,n}(x) < \frac{1}{2^n}$. So $t_n \nearrow f$ pointwise. $\square$

Lemma 11.5 (Tietze's Extension Lemma)**.** If $C \subseteq X$ is closed and $g : C \to \mathbb{R}$ is continuous, then there exists $G : X \to \mathbb{R}$ such that G is continuous and $G|_C = g$.

Theorem 11.6 (Lusin's Theorem)**.** Let (X, Σ, μ) be a measure space with X compact and μ finite regular and $\Sigma \supseteq \mathcal{B}(X)$. If $f : X \to \mathbb{R}$ is measurable, then for every $\epsilon > 0$, there exists a continuous $F : X \to \mathbb{R}$ such that $\mu(\{x \in X \mid f(x) \ne F\}) < \epsilon$.

Proof. Suppose f is bounded. Without loss of generality, say $f : X \to [0, 1)$. Fix $\epsilon > 0$. For $n, k \in \mathbb{N}$, with $0 \le k < 2^n$, define $A_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right)$. By regularity of μ, we can find compact $K_{n,k} \subseteq A_{n,k}$ with $\mu(A_{n,k} - K_{n,k}) < \frac{\epsilon}{2^{2k}}$. Define $C_n = \bigcup_{k=1}^\infty K_{n,k}$. Then $\mu(X - C_n) < \frac{\epsilon}{2^k}$. Define $g_n : C \to \mathbb{R}$ by $g_n(x) = \sum_{k=0}^{2^n-1} \frac{k}{2^n} \chi_{C_{n,k}}$. Note that g_n is continuous on C (since C_n is locally compact).

Now, define $C = \bigcap_{n=1}^{\infty} C_n$. Then $\mu(X - C) < \sum_{n=1}^{\infty} \mu(X - C_n) < \epsilon$. Furthermore, $C \subseteq C_n$ for all n, so that $|g_n - f| < \frac{1}{2^n}$ on C. Thus, on C, $g_n \to f$ uniformly. Hence, $f|_C$ is continuous. By Lemma 11.5, we can extend each $f|_C$ to a continuous $F : X \to \mathbb{R}$ such that $F|_C - f|_C$. Therefore, $F = f$ on C, and $\mu(\{x \in X \mid f(x) \neq F(x)\}) \leq \mu(X - C) < \epsilon$.

Suppose f is not bounded. Then consider $g = \tan^{-1}(f)$. Then g is bounded, so we can find continuous $G : X \to \infty$ such that $\mu(\{x \in X \mid g(x) \neq G(x)\}) < \epsilon$. Since arctangent is bijective and continuous, it follows that

$$\begin{aligned}\mu(\{x \in X \mid f(x) \neq \tan(G(x))\} &= \mu(\{x \in X \mid \tan(g(x)) \neq \tan(G(x))\} \\ &= \mu(\{x \in X \mid g(x) \neq G(x)\}) < \epsilon.\end{aligned}$$

Thus, choosing $F(x) = \tan(G(x))$, the result follows. $\square$

12 September 25, 2013

Today, our goal is to construct the Lebesgue integral. As usual, let (X, Σ, μ) be a measure space. All functions that we work with are measurable.

Definition 12.1 (Lebesgue Integral for Simple Functions)**.** Let $s : X \to [0, \infty)$ be simple. Let $s(X) = \{a_1, \ldots, a_n\}$ for $a_i \in \mathbb{R}$ and $n \in \mathbb{N}$. Let $A_i = s^{-1}(\{a_i\})$. Then define

$$\int_X s\, d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

This is the *Lebesgue integral* of simple function s. Note that this sum is well defined since s was assumed to be positive.

Remark 12.2 (Monotonicity)**.** If s, t are simple and $0 \leq s \leq t$, then

$$\int_X s\, d\mu \leq \int_X t\, d\mu.$$

Definition 12.3 (Lebesgue Integral for Positive Functions)**.** Let $f : X \to [0, \infty]$ be measurable. Define the *Lebesgue integral of f over X with respect to μ* as

$$\int_X f\, d\mu = \sup_{\substack{0 \leq s \leq f \\ s \text{ simple}}} \int_X s\, d\mu.$$

Remark 12.4. If s is simple and $s(X) = \{a_1, \ldots, a_n\}$ and $A_i = s^{-1}(\{a_i\})$, then

$$\sum_{i=1}^{n} a_i \mu(A_i) = \sup_{\substack{0 \leq t \leq s \\ t \text{ simple}}} \int_X t\, d\mu.$$

Proof. Let $0 \leq t \leq s$ be simple. Then $\int_X t\, d\mu \leq \int_X s\, d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$ by Remark 12.2. Since t was arbitrary, it follows that

$$\sup_{\substack{0 \leq t \leq s \\ t \text{ simple}}} \int_X t\, d\mu \leq \sum_{i=1}^{n} a_i \mu(A_i).$$

By choosing $t = s$, equality holds. $\square$

Definition 12.5 (Lebesgue Integral)**.** Say $f : X \to [-\infty, \infty]$ is measurable. Let $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Then the *Lebesgue integral of f over X with respect to μ* is

$$\int_X f\, d\mu = \int_X f^+\, d\mu - \int_X f^-\, d\mu,$$

provided that $\int_X f^+\, d\mu < \infty$ or $\int_X f^-\, d\mu < \infty$.

Example 12.6. Let $X = \mathbb{N}$ and μ be the counting measure and $\Sigma = \mathcal{P}(\mathbb{N})$. Given $a : \mathbb{N} \to \mathbb{R}$,

1. if $\sum_{i=1}^{\infty} |a(n)| < \infty$, then $\sum_{i=1}^{\infty} a(n) = \int_{\mathbb{N}} a \, d\mu$;
2. if $\sum_{i=1}^{\infty} a(n)$ is conditionally convergent, then $\int_{\mathbb{N}} a \, d\mu$ is not defined.

Remark 12.7 (Monotonicity)**.** If $0 \leq f \leq g$, clearly $\int_X f \, d\mu \leq \int_X g \, d\mu$.

Proposition 12.8 (Monotone Convergence Theorem)**.** Let $\{f_n\}$ be a sequence of measurable functions such that $0 \leq f_n \leq f_{n+1}$. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Then

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Proof. Since $f_n \leq f$, by monotonicity, $\int_X f_n \, d\mu \leq \int_X f \, d\mu$, so that

$$\lim_{n\to\infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu.$$

Now, let $0 \leq s \leq f$ be simple. Fix $\epsilon > 0$. Let $E_n = \{x \in X \mid f_n \geq (1-\epsilon)s\}$. Since $f_n \leq f_{n+1}$, $E_n \subseteq E_{n+1}$.

If $f(x) = 0$, then $s(x) = 0$ and $f_n(x) = 0$. So $x \in E_n$ for all $n \in \mathbb{N}$. If $f(x) > 0$, then $(1-\epsilon)s(x) \leq f(x)$, so we can find $N \in \mathbb{N}$ such that for $n \geq N$, $f_n(x) \geq (1-\epsilon)s(x)$. Thus $x \in E_n$ for $n \geq N$. It follows that $\bigcup_{n=1}^{\infty} E_n = X$.

Let $s(\{X\}) = \{a_1, \ldots, a_m\}$ and $A_i = s^{-1}(\{a_i\})$. Then

$$\int_X (1-\epsilon)s\chi_{E_n} \, d\mu = \sum_{i=1}^{m} (1-\epsilon)a_i\mu(A_i \cap E_n)$$

$$\Rightarrow \lim_{n\to\infty} \int_X (1-\epsilon)s\chi_{E_n} \, d\mu = \sum_{i=1}^{m} (1-\epsilon)a_i\mu(A_i) = \int_X (1-\epsilon)s \, d\mu.$$

By definition of E_n, $f_n \geq (1-\epsilon)s_n\chi_{E_n}$, so that

$$\lim_{n\to\infty} \int_X f_n \, d\mu \geq \lim_{n\to\infty} \int_X (1-\epsilon)s_n\chi_{E_n} \, d\mu = \int_X (1-\epsilon)s \, d\mu.$$

Taking $\epsilon \to 0$, then the supremum over simple functions $0 \leq s \leq f$, it follows that

$$\lim_{n\to\infty} \int_X f_n \, d\mu \geq \sup_{\substack{0 \leq s \leq f \\ s \text{ simple}}} \int_X s \, d\mu \geq \int_X f \, d\mu.$$

This proves the theorem ☐

Note 12.9. Monotone Converge fails without the assumption that $f_n \geq 0$. As an example, consider $f_n(x) = -\frac{1}{n}$. Certainly $f_n \to f = 0$. For all n, $\int_X f_n \, d\mu = -\infty$, but $\int_X f \, d\mu = 0$.

13 September 27, 2013

Definition 13.1 (Lebesgue 1 Space)**.** Let

$$\mathcal{L}^1(X) = \left\{ f : X \to [-\infty, \infty] \,\middle|\, f \text{ meas, } \int_X f^+ \, d\mu < \infty, \int_X f^- \, d\mu < \infty \right\}.$$

$\mathcal{L}^1(X)$ is called the *Lebesgue 1 Space.*

Proposition 13.2 (Linearity of Lebesgue Integrals)**.** Let $f, g \in \mathcal{L}^1(X)$ and let $\alpha, \beta \in \mathbb{R}$. Then

$$\int_X (\alpha f + \beta g) \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu.$$

More generally, if f and g are Lebesgue integrable in the extended sense and $\int_X \alpha f \, d\mu + \int_X \beta g \, d\mu$ is defined, then $\alpha f + \beta g$ is Lebesgue integrable in the extended sense and the same formula applies.

Proof. Without loss of generality, we may assume $\alpha, \beta > 0$. Surely, if f and g are non-negative and simple, this is true (this can easily be checked by chasing the definition of simple functions and their integrals).

Suppose $f, g \geq 0$. Then by Proposition 11.4, we can find simple functions s_n and t_n such that $s_n, t_n \geq 0$ and $s_n \nearrow f$ and $t_n \nearrow g$. Then $(s_n + t_n) \nearrow f + g$ pointwise. By Monotone Convergence, it follows that

$$\lim_{n\to\infty} \int_X s_n \, d\mu + \lim_{n\to\infty} \int_X t_n \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$$

and

$$\lim_{n\to\infty} \int_X (s_n + t_n) \, d\mu = \int_X (f + g) \, d\mu.$$

Since the left-hand side limits are equal, it follows that

$$\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu.$$

Now consider arbitrary f and g. We can write $f = f^+ - f^-$ and $g = g^+ - g^-$. Let $P = f^+ + g^+$ and $N = f^- + g^-$. So $f + g = P - N$. Let $H = f + g$. By definition,

$$\int_X H \, d\mu = \int_X H^+ \, d\mu - \int_X H^- \, d\mu.$$

Since $H = H^+ - H^- = P - N$, we have $H^+ + N = H^- + P$. Since we showed linearity holds for non-negative functions, we have

$$\int_X H^+ \, d\mu + \int_X N \, d\mu = \int_X H^- \, d\mu + \int_X P \, d\mu.$$

This implies

$$\int_X H\,d\mu = \int_X H^+\,d\mu - \int_X H^-\,d\mu = \int_X P\,d\mu - \int_X N\,d\mu.$$

Again, by linearity for non-negative functions, $\int_X P\,d\mu = \int_X f^+\,d\mu + \int_X g^+\,d\mu$ and $\int_X N\,d\mu = \int_X f^-\,d\mu + \int_X g^-\,d\mu$. Thus, we have

$$\int_X (f+g)\,d\mu = \int_X f^+\,d\mu + \int_X g^+\,d\mu - \int_X f^-\,d\mu - \int_X g^-\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu.$$

$\square$

Theorem 13.3 (Beppo-Levi's Theorem)**.** If $f_n \geq 0$ is measurable for each $n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} \int_X f_n\,d\mu = \int_X \sum_{n=1}^{\infty} f_n\,d\mu.$$

Proof. Let $S_n = \sum_{k=1}^n f_k$. Certainly $S_n \geq 0$ and is measurable and $S_n \nearrow \sum_{k=1}^{\infty} f_k$. The result follows from Monotone Convergence. $\square$

Corollary 13.4. Let

$$f(x) = \sum_{m,n\in\mathbb{N}} \chi_{\{|x-\frac{m}{n}|<1\}} \frac{1}{\left|x-\frac{m}{n}\right|^{\frac{1}{2}}} \cdot \frac{1}{2^{m+n}}.$$

Then $f(x) < \infty$ a.e.

Proof. By Beppo-Levi, it follows that $\int_{\mathbb{R}} f\,d\lambda < \infty$. This implies that f is finite almost everywhere. $\square$

Corollary 13.5. There exists a measurable $f : \mathbb{R} \to [0, \infty]$ that is finite a.e. such that for every $a < b$,

$$\int_a^b f\,d\lambda = \infty$$

Proof. Define

$$g(x) = \sum_{m,n\in\mathbb{N}} \chi_{\{|x-\frac{m}{n}|<1\}} \frac{1}{\left|x-\frac{m}{n}\right|^{\frac{1}{2}}} \cdot \frac{1}{2^{m+n}}.$$

By Corollary 13.4, $g(x) < \infty$ a.e. Choose $f = g^2$. Then certainly f is finite a.e. as well. However, it is clear that

$$\int_a^b f\,d\lambda = \infty$$

for every $a < b$. $\square$

14 September 30, 2013

As usual, let (X, Σ, μ) be a measure space. Unless otherwise specified, all functions in consideration will map from X to $\mathbb{R}$ (or $[-\infty, \infty]$). We would like to have something of the form

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu,$$

where $f_n \to f$ pointwise. We already have some version of this, namely Monotone Convergence. However, in general, this is not true.

Example 14.1 (Mass escaping to ∞)**.** Let $X = \mathbb{R}$ and $\mu = \lambda$. Let $f_n = \chi_{[n,n+1]}$. Certainly, $f_n \to 0$ pointwise, but

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\lambda = 1 \neq 0 = \int_{\mathbb{R}} 0 \, d\lambda.$$

Example 14.2 (Mass collecting at a point)**.** Let $X = \mathbb{R}$ and $\mu = \lambda$. Let $f_n = n\chi_{\left[0,\frac{1}{n}\right]}$. Certainly, $f_n \to 0$ pointwise, but

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\lambda = 1 \neq 0 = \int_{\mathbb{R}} 0 \, d\lambda.$$

Fortunately, we can show that the desired result holds under a few conditions.

Lemma 14.3 (Fatou's Lemma)**.** Let $f_n \geq 0$ be measurable for $n \in \mathbb{N}$. Then

$$\liminf_{n\to\infty} \int_X f_n \, d\mu \geq \int_X \liminf_{n\to\infty} f_n \, d\mu.$$

Proof. Let $g_n = \inf_{k\geq n} f_k$. Then $g_n \to \liminf_{k\to\infty} f_k$ as $n \to \infty$. Further, it is clear that $g_n \leq g_{n+1}$ and $g_n \geq 0$ for each $n \in \mathbb{N}$. Thus, by Monotone Convergence,

$$\int_X \liminf_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_X g_n \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu,$$

where the last inequality comes from the fact that $f_n \geq g_n$. $\square$

Theorem 14.4 (Dominated Convergence)**.** Let f_n for $n \in \mathbb{N}$ and f be measurable such that $f_n \to f$ pointwise. Suppose there exists a measurable $G \in \mathcal{L}^1(X)$ with $G \geq 0$ such that $|f_n| \leq G$ for all n. Then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Proof. Let $g_n = G + f_n$. Then $g_n \geq 0$. Further, we certainly have $g_n \to G + f$ pointwise. Therefore, by Fatou's lemma,

$$\liminf_{n\to\infty} \int_X g_n \, d\mu \geq \int_X (G + f) \, d\mu. \tag{1}$$

Note since $G \in \mathcal{L}^1(X)$ and $G \geq 0$, $\int_X G\, d\mu < \infty$, so that

$$\liminf_{n\to\infty} \int_X g_n\, d\mu = \liminf_{n\to\infty} \int_X (G + f_n)\, d\mu = \int_X G\, d\mu + \lim_{n\to\infty} \int_X f_n\, d\mu$$

where the last equality follows from linearity of Lebesgue integrals. So, by Equation 1 it follows that

$$\int_X G\, d\mu + \liminf_{n\to\infty} \int_X f_n\, d\mu \geq \int_X G\, d\mu + \int_X f\, d\mu \Rightarrow \liminf_{n\to\infty} \int_X f_n\, d\mu \geq \int_X f\, d\mu.$$

Let $h_n = G - f_n$. Similarly, $h_n \geq 0$ and $h_n \to G - f$ pointwise. Therefore, by Fatou's lemma,

$$\liminf_{n\to\infty} \int_X h_n\, d\mu \geq \int_X (G - f)\, d\mu. \tag{2}$$

As before, we have

$$\liminf_{n\to\infty} \int_X h_n\, d\mu = \liminf_{n\to\infty} \int_X (G - f_n)\, d\mu = \int_X G\, d\mu - \limsup_{n\to\infty} \int_X f_n\, d\mu$$

So, by Equation 2 it follows that

$$\int_X G\, d\mu - \limsup_{n\to\infty} \int_X f_n\, d\mu \geq \int_X G\, d\mu - \int_X f\, d\mu,$$

or

$$\limsup_{n\to\infty} \int_X f_n\, d\mu \leq \int_X f\, d\mu.$$

This implies that $\liminf_{n\to\infty} \int_X f_n\, d\mu = \limsup_{n\to\infty} \int_X f_n\, d\mu$. Thus, it follows that $\lim_{n\to\infty} \int_X f_n\, d\mu$ exists and is equal to $\int_X f\, d\mu$. □

Definition 14.5 (Laplace Transform)**.** Let $X = [0, \infty)$ and $\mu = \lambda$. Define the *Laplace transform* of f by

$$\mathcal{L}f(s) = \int_0^\infty e^{-st} f(t) dt.$$

Proposition 14.6. Say $f : [0, \infty) \to \mathbb{R}$ is such that $t|f(t)| \in \mathcal{L}^1([0, \infty))$. Then the Laplace transform of f is differentiable, and $\frac{d}{ds}\mathcal{L}f = \mathcal{L}(-tf(t))$.

Proof. We have

$$\frac{d}{ds}\mathcal{L}f(s) = \lim_{h\to 0} \int_0^\infty \frac{e^{-(s+h)t} - e^{-st}}{h} f(t)\, dt.$$

By the Mean Value Theorem, $\left|\frac{e^{-(s+h)t} - e^{-st}}{h}\right| \leq t$, so $\left|\frac{e^{-(s+h)t} - e^{-st}}{h} f(t)\right| \leq t|f(t)|$. Therefore, by Dominated Convergence,

$$\begin{aligned}\frac{d}{ds}\mathcal{L}f(s) &= \int_0^\infty \lim_{h\to 0} \frac{e^{-(s+h)t} - e^{-st}}{h} f(t)\, dt \\ &= \int_0^\infty \frac{d}{ds}\left(e^{-st}\right) f(t)\, dt = \int_0^\infty -te^{-st} f(t)\, dt = \mathcal{L}(-tf(t))(s).\end{aligned}$$

□

15 October 2, 2013

Today, we will introduce several different notions of convergence of functions. As usual, we let (X, Σ, μ) be a measure space, and let $f_n, f : X \to \mathbb{R}$ be measurable for $n \in \mathbb{N}$

Definition 15.1 (Convergence Almost Everywhere)**.** We say $f_n \to f$ μ-a.e. if there exists a μ-null set $N \subseteq X$ such that $f_n \to f$ pointwise on N^c.

Definition 15.2 (Convergence in Measure)**.** We say $f_n \to f$ in μ-measure if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mu\left(\{x \in X \,|\, |f_n - f| > \epsilon\}\right) = 0.$$

Definition 15.3 (Convergence in $\mathcal{L}^p$)**.** For any $p \geq 1$, we say $f_n \to f$ in $\mathcal{L}^p$ (we will define $\mathcal{L}^p$ for $p > 1$ a bit later) if

$$\lim_{n\to\infty} \left(\int_X |f_n - f|^p \, d\mu \right)^{\frac{1}{p}} = 0.$$

We will see how these various notions of convergence relate to each other. In generate, convergence almost everywhere does not imply convergence in measure.

Example 15.4 (Mass escaping to ∞)**.** Let $X = \mathbb{R}$, $\mu = \lambda$, and $f_n = \chi_{[n,n+1)}$. Then $f_n \to f = 0$ λ-a.e. (in fact, pointwise everywhere), but certainly $f_n \not\to f$ in λ-measure, since $\mu\left(\{x \in X \,|\, |f_n - f| > \epsilon\}\right) = 1$ for all $n \in \mathbb{N}$ and $\epsilon < 1$.

However, if the space is finite, then mass cannot escape.

Theorem 15.5 (Egorov's Theorem)**.** Say $\mu(X) < \infty$ and $f_n \to f$ μ-a.e. Then for every $\epsilon > 0$, there exists $A_\epsilon \in \Sigma$ such that $\mu(A_\epsilon) < \epsilon$ and $f_n \to f$ uniformly on A_ϵ^c.

Proof. Let $\epsilon > 0$. Define for each $N, k \in \mathbb{N}$

$$C_{N,k} = \left\{ x \in X \,\middle|\, \exists\, n \geq N \text{ such that } |f_n - f| \geq \frac{1}{k} \right\}.$$

Fix $k \in \mathbb{N}$. Certainly $C_{N,k} \subseteq C_{N+1,k}$. Further, note

$$\bigcap_{N\in\mathbb{N}} C_{N,k} \subseteq \{x \in X \,|\, f_n \not\to f \text{ pointwise}\}.$$

This implies that $\mu\left(\bigcap_{N\in\mathbb{N}} C_{N,k}\right) = 0$. Since $\mu(X) < \infty$, by Homework 1, this implies that $\lim_{N\to\infty} \mu(C_{N,k}) = 0$. So for each k, we can find $N_k \in \mathbb{N}$ such that $\mu(C_{N_k,k}) < \frac{\epsilon}{2^k}$.

Let $A = \bigcup_{k\in\mathbb{N}} C_{N_k,k}$. Then $\mu(A) < \epsilon$ and for each $k \in \mathbb{N}$,

$$A^c \subseteq C_{N_k,k}^c = \left\{ x \in X \,\middle|\, \forall\, n \geq N_k,\ |f_n - f| < \frac{1}{k} \right\}.$$

Therefore, on A^c, $|f_n - f| < \frac{1}{k}$ for all $n \geq N_k$. This implies that $f_n \to f$ uniformly on A^c. $\square$

Corollary 15.6. Suppose $\mu(X) < \infty$ and $f_n \to f$ μ-a.e. Then $f_n \to f$ in μ-measure.

Proof. Fix $\epsilon > 0$. By Egorov's theorem, there is an $A \in \Sigma$ such that $\mu(A) < \delta$ and $f_n \to f$ uniformly on A^c. So for large n, $\{x \in X \,|\, |f_n - f| \leq \epsilon\} \supseteq A^c$. Hence, $\{x \in X \,|\, |f_n - f| > \epsilon\} \subseteq A$, so that

$$\mu\left(\{x \in X \,|\, |f_n - f| < \epsilon\}\right) \leq \mu(A) < \delta.$$

Taking $\delta \to 0$, the result follows. $\square$

In general, convergence in measure does not imply convergence almost everywhere.

Example 15.7. Let $f_1 = \chi_{[0,\frac{1}{2})}$, $f_2 = \chi_{[\frac{1}{2},1)}$, $f_3 = \chi_{[0,\frac{1}{4})}$, $f_4 = \chi_{[\frac{1}{4},\frac{1}{2})}$, and so on. Certainly $f_n \to f$ in λ-measure. However, $f_n(x) \not\to 0$ for every $x \in [0, 1)$.

16 October 4, 2013

Theorem 16.1. If $f_n \to f$ in μ-measure, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ μ-a.e.

Proof. We can find n_1 such that $\mu\left(\{x \in X \,|\, |f_{n_1} - f| > 1\}\right) < 1$ for all n. Given $n_1, \ldots, n_k$, we can find $n_{k+1} > n_k$ such that

$$\mu\left(\left\{x \in X \,\middle|\, |f_{n_{k+1}} - f| > \frac{1}{k+1}\right\}\right) < \frac{1}{2^{k+1}}.$$

Let $A_k := \left\{x \in X \,\middle|\, |f_{n_k} - f| > \frac{1}{k}\right\}$.

Let $B = \{x \in X \,|\, x \text{ belongs to finitely many } A_k\text{'s}\}$. Note that $f_{n_k} \to f$ pointwise on B. Further, note that

$$\begin{aligned} B^c &= \{x \in X \,|\, x \text{ belongs to infinitely many } A_k\text{'s}\} \\ &= \{x \in X \,|\, \forall\, m \in \mathbb{N},\, \exists\, n \geq m \text{ such that } x \in A_m\} \\ &= \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} A_n. \end{aligned}$$

However, $\mu\left(\bigcup_{n \geq m} A_n\right) < \sum_{n \geq m} \mu(A_n) < \frac{1}{2^{m-1}}$. Therefore,

$$\mu(B^c) = \lim_{m \to \infty} \mu\left(\bigcup_{n \geq m} A_n\right) = 0.$$

□

Definition 16.2 (Normed Vector Space)**.** X is a *normed vector space* over $\mathbb{R}$ if

1. X is a vector space over $\mathbb{R}$;
2. X is endowed with a norm $\|\cdot\|$ that satisfies:
    a. for every $x \in X$, $\|x\| \in [0, \infty)$;
    b. $\|x\| = 0$ if and only if $x = 0$;
    c. for every $\alpha \in \mathbb{R}$ and $x \in X$, $\|\alpha x\| = \alpha \|x\|$;
    d. for every $x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$.

Definition 16.3 (Banach Space)**.** A normed vector space X is a *Banach space* if it is Cauchy complete under the metric $d(x, y) = \|x - y\|$.

Definition 16.4 ($\mathcal{L}^p$ space)**.** For $p \in [1, \infty)$, then the $\mathcal{L}^p$ *space* of complete measure space (X, Σ, μ) is

$$\mathcal{L}^p(X, \Sigma, \mu) = \left\{f : X \to [-\infty, \infty] \,\middle|\, f \text{ is measurable and } \int_X |f|^p \, d\mu < \infty\right\}.$$

For every $f \in \mathcal{L}^p$, define $\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}$.

Definition 16.5 ($\mathcal{L}^\infty$ space)**.** For a measurable f, define

$$\begin{aligned}\|f\|_\infty &= \sup\{\lambda \mid \mu(\{x \in X \mid |f(x)| > \lambda\}) > 0\}\\ &= \inf\{\lambda \mid \mu(\{x \in X \mid |f(x)| > \lambda\}) = 0\}\end{aligned}$$

Define $\mathcal{L}^\infty(X,\mu) = \{f \mid \|f\|_\infty < \infty\}$.

Definition 16.6. If f and g are measurable, then we can define an equivalence relation $\sim$ such that $f \sim g$ if and only if $f = g$ a.e. Subsequently, we will only work with complete measure spaces.

Note 16.7. Say $f, g \in \mathcal{L}^p$. Then $f \sim g$ if and only if $\int_A f\,d\mu = \int_A g\,d\mu$ for every $A \in \Sigma$ if and only if $\int_X fh\,d\mu = \int_X gh\,d\mu$ for every bounded and measurable h.

Definition 16.8 (L^p space)**.** For $p \in [1,\infty]$, define $L^p(X,\mu) = \mathcal{L}^p(X,\mu)/\sim$, the set of equivalence classes of $\mathcal{L}^p$ under the equivalence relation $\sim$. For a class $[f] \in L^p(X,\mu)$, pick a member function $f \in [f]$ and define $\|[f]\|_p = \|f\|_p$.

Note 16.9. For our purposes, we will treat members of L^p as functions, not equivalence classes, but we cannot talk about the value of those functions at a point. But we can talk about the integral of the function over a set, or talk about how it compares with another function a.e.

We would like to show that L^p is a Banach space eventually. We do this through several results.

17 October 7, 2013

Remark 17.1. Note $|f| \leq \|f\|_\infty$ a.e. Further $f_n \to f$ in L^∞ if and only if $\|f_n - f\| \to 0$ if and only if $f_n \to f$ uniformly a.e.

Lemma 17.2 (Young's Inequality)**.** If $x, y \in \mathbb{R}$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q \in (1, \infty)$, then

$$|xy| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}.$$

Proof. We have $\ln(|x||y|) = \frac{\ln |x|^p}{p} + \frac{\ln |y|^q}{q}$. By concavity of logarithms and Jensen's inequality,

$$\frac{\ln |x|^p}{p} + \frac{\ln |x|^q}{q} \leq \ln\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right).$$

Since ln is an increasing function, it follows that $|xy| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}$. □

Theorem 17.3 (Hölder's Inequality)**.** If $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

Proof. Without loss of generality, we may assume $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$.

Suppose $p, q \in (1, \infty)$. Define $\widetilde{f} = \frac{f}{\|f\|_p}$ and $\widetilde{g} = \frac{g}{\|g\|_q}$. Note $\|\widetilde{f}\|_p = \|\widetilde{g}\|_q = 1$. Then by Young's inequality,

$$\int_X |\widetilde{f}\widetilde{g}| \, d\mu \leq \int_X \left(\frac{|\widetilde{f}|^p}{p} + \frac{|\widetilde{g}|^q}{q}\right) d\mu \leq \frac{\|\widetilde{f}\|_p^p}{p} + \frac{\|\widetilde{g}\|_q^q}{q} = 1.$$

Hence,

$$\|fg\|_1 = \int_X |fg| \, d\mu = \|f\|_p \|g\|_q \int_X |\widetilde{f}\widetilde{g}| \, d\mu \leq \|f\|_p \|g\|_q .$$

Suppose $p = 1$ and $q = \infty$ (the case where $p = \infty$ and $q = 1$ is similar). Then $|g| \leq \|g\|_\infty$ a.e., so that $|fg| \leq |f| \|g\|_\infty$ a.e., so that

$$\|fg\|_1 = \int_X |fg| \, d\mu \leq \int_X |f| \|g\|_\infty \, d\mu = \|g\|_\infty \int_X |f| \, d\mu = \|g\|_\infty \|f\|_1 .$$

□

Corollary 17.4 (Hölder's Inequality)**.** If $p_i, q \in [1, \infty]$ for $1 \leq i \leq N$ with $\sum_{i=1}^N \frac{1}{p_i} = \frac{1}{q}$, and if $f_i \in L^{p_i}$, then

$$\left\|\prod_{i=1}^N f_i\right\|_q \leq \prod_{i=1}^N \|f_i\|_{p_i} .$$

Lemma 17.5 (Duality Equality)**.** If $p \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|f\|_p = \sup_{g \in L^q - \{0\}} \frac{1}{\|g\|_q} \int_X fg \, d\mu.$$

Proof. Hölder's inequality implies that for all $g \in L^q$,

$$\int_X fg \, d\mu \leq \|f\|_p \|g\|_p \Rightarrow \|f\|_p \geq \frac{1}{\|g\|_q} \int_X fg \, d\mu.$$

Taking the supremum over $L^q - \{0\}$, we have

$$\|f\|_p \geq \sup_{g \in L^q - \{0\}} \frac{1}{\|g\|_q} \int_X fg \, d\mu.$$

It suffices to show there exists a $g \in L^q - \{0\}$ such that equality holds. Choose $g = \frac{1}{\|f\|_p^{p-1}} f^{p-1} \operatorname{sign}(f)$. Then

$$\|g\|_q^q = \int_X |g|^q \, d\mu = \frac{1}{\|f\|_p^{(p-1)q}} \int_X |f|^{(p-1)q} \, d\mu = \frac{1}{\|f\|_p^p} \int_X |f|^p \, d\mu = \frac{\|f\|_p^p}{\|f\|_p^p} = 1.$$

So $g \in L^q$, $\|g\|_q = 1$, and

$$\frac{1}{\|g\|_q} \int_X fg \, d\mu = \int_X \frac{|f|^p}{\|f\|_p^{p-1}} \, d\mu = \|f\|_p \, .$$

□

Corollary 17.6 (Duality Equality)**.** If $p \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|f\|_p = \sup_{\substack{g \in L^q - \{0\} \\ g \text{ simple}}} \frac{1}{\|g\|_q} \int_X fg \, d\mu.$$

Proof. This follows from Lemma 17.5 and the density of simple functions in L^p. □

Before, we go on, let us motivate Hölder's inequality through a dimension counting / scaling argument.

Note 17.7. Choosing $X = \mathbb{R}^d$, $\mu = \lambda$, and $\alpha > 0$, set $f_\alpha(x) = f\left(\frac{x}{\alpha}\right)$ and $g_\alpha(x) = g\left(\frac{x}{\alpha}\right)$. Then

$$\int_{\mathbb{R}^d} f_\alpha g_\alpha \, d\lambda = \int_{\mathbb{R}^d} f\left(\frac{x}{\alpha}\right) g\left(\frac{x}{\alpha}\right) \, d\lambda = \alpha^d \int_{\mathbb{R}^d} fg \, d\lambda.$$

We also have

$$\|f_\alpha\|_p = \left(\int_{\mathbb{R}^d} \left| f\left(\frac{x}{\alpha}\right)\right|^p d\lambda\right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^d} \alpha^d |f(x)|^p \, d\lambda\right)^{\frac{1}{p}} = \alpha^{\frac{d}{p}} \|f\|_p \, .$$

Similarly, $\|g_\alpha\|_p = \alpha^{\frac{d}{q}} \|g\|_q$. So $\|f_\alpha\|_p \|g_\alpha\|_p = \alpha^{\frac{d}{p} + \frac{d}{q}} \|f\|_p \|g\|_p$. This implies that we should have $1 = \frac{1}{p} + \frac{1}{q}$.

18 October 9, 2013

Theorem 18.1 (Minkowski's Inequality)**.** Let $p \in [1,\infty]$. If $f, g \in L^p$, then $f+g \in L^p$ and $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

Proof. Suppose $p \in [1,\infty)$. By the Duality equality,

$$\begin{aligned}
\|f+g\|_p &= \sup_{h \in L^q - \{0\}} \frac{1}{\|h\|_q} \int_X (f+g)h \, d\mu \\
&\le \sup_{h \in L^q - \{0\}} \frac{1}{\|h\|_q} \int_X fh \, d\mu + \sup_{h \in L^q - \{0\}} \frac{1}{\|h\|_q} \int_X gh \, d\mu \\
&= \|f\|_p + \|g\|_p .
\end{aligned}$$

Suppose $p = \infty$. Then

$$\begin{aligned}
\|f+g\|_p &= \sup \{\lambda \mid \mu(\{x \in X \mid |f(x)+g(x)| > \lambda\}) > 0\} \\
&= \sup \{\lambda \mid \mu(\{x \in X \mid |f(x)| > \lambda\}) > 0\} \\
&\qquad + \sup \{\lambda \mid \mu(\{x \in X \mid |g(x)| > \lambda\}) > 0\} \\
&= \|f\|_p + \|g\|_p .
\end{aligned}$$

□

Lemma 18.2 (Countable Triangle Inequality)**.** Let $f_n \in L^p$ for $p \in [1,\infty]$. Say $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$. Then

1. there exists $f \in L^p$ such that $f = \sum_{n=1}^{\infty} f_n$;
2. $\sum_{n=1}^{\infty} f_n \to f$ a.e. and $\sum_{n=1}^{\infty} f_n \to f$ in L^p;
3. $\|\sum_{n=1}^{\infty} f_n\|_p \le \sum_{n=1}^{\infty} \|f_n\|_p$.

Proof. Suppose $p \in [1,\infty)$. Let $F(x) = \sum_{n=1}^{\infty} |f_n(x)|$. Let $t_N = \sum_{n=1}^{N} |f_n|$ and $s_N = \sum_{n=1}^{N} f_n$.

Note $t_N^p \to F^p$ as $N \to \infty$, $t_N^P \le t_{N+1}^P$ and $0 \le t_N^P$. By Monotone Convergence

$$\int_X F^p \, d\mu = \lim_{N\to\infty} \int_X t_N^p \, d\mu.$$

Hence

$$\|F\|_p = \left(\int_X F^p \, d\mu\right)^{\frac{1}{p}} = \lim_{N\to\infty} \|t_N\|_p \le \lim_{N\to\infty} \sum_{n=1}^{N} \|f_n\|_p = \sum_{n=1}^{\infty} \|f_n\|_p < \infty.$$

Hence $F \in L^p$. Hence $\sum_{n=1}^{\infty} |f_n| = F < \infty$ a.e., which implies that $\sum_{n=1}^{\infty} f_n$ is absolutely convergent a.e. Let $f = \sum_{n=1}^{\infty} f_n$. Then $\sum_{n=1}^{N} f_n \to f$ a.e. as $N \to \infty$. By the triangle inequality, $\left|\sum_{n=1}^{N} f_n\right| \le \sum_{n=1}^{N} |f_n|$. So, taking $N \to \infty$, it follows that $|f| \le F \in L^p$, which implies that $f \in L^p$. This proves 1.

Since $|f| \leq F$, $\|f\|_p \leq \|F\|_p \leq \sum_{n=1}^{\infty} \|f_n\|_p$, which proves 3. Finally, using 3,

$$\left\| f - \sum_{n=1}^{\infty} f_n \right\|_p = \left\| \sum_{n=N+1}^{\infty} f_n \right\|_p \leq \sum_{n=N+1}^{\infty} \|f_n\|_p \to 0$$

as $N \to \infty$. This proves 2.

Suppose $p = \infty$. Then $|f_n| \leq \|f_n\|_\infty$ a.e. Hence, we can find a null set M such that for every $x \in M^c$, $|f_n(x)| < \|f_n\|_\infty$ for every $n \in \mathbb{N}$. Hence, for $x \in \mathbb{N}$, $\sum_{n=1}^{\infty} f_n(x)$ is absolutely convergent. Hence, we can define $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for $x \in M^c$ (and we can let $f = 0$ on M).

By the triangle inequality, $\left|\sum_{n=1}^{N} f_n\right| \leq \sum_{n=1}^{N} |f_n|$. Taking $N \to \infty$, it follows that $|f| \leq \sum_{n=1}^{\infty} |f_n| \leq \sum_{n=1}^{\infty} \|f_n\|_\infty$ on M^c. So $f \in L^\infty$. Further, since M is null, this implies that $\|f\|_\infty \leq \sum_{n=1}^{\infty} \|f_n\|_\infty$. This proves 1 and 3.

Finally, applying part 3, as $N \to 0$.

$$\left\| f - \sum_{n=1}^{N} f_n \right\|_\infty = \|sum_{n=N+1} f_n\|_\infty \leq \sum_{n=N+1}^{\infty} \|f_n\|_\infty \to 0.$$

This proves 2. ☐

We can finally prove the completeness of L^p spaces.

Proposition 18.3 (L^p is Complete)**.** Let $\{f_n\}$ be a Cauchy sequence in L^p. Then there exists $f \in L^p$ such that $f_n \to f$ in L^p.

Proof. It is enough to show that there exists a subsequence $\{f_{n_k}\}$ such that f_{n_k} is convergent in L^p. Choose $n_1 = 1$. For each k, we can choose $n_{k+1} > n_k$ such that $\left\|f_{n_{k+1}} - f_{n_k}\right\|_p < \frac{1}{2^k}$. Then let $f = f_{n_1} + \sum_{k=1}^{\infty} \left(f_{n_{k+1}} - f_{n_k}\right)$. Since $\sum_{k=1}^{\infty} \left\|f_{n_{k+1}} - f_{n_k}\right\|_p < \infty$, by Lemma 18.2, $\sum_{k=1}^{\infty} \left(f_{n_{k+1}} - f_{n_k}\right) \in L^p$ and is convergent in L^p. Observe $f_{n_1} + \sum_{k=1}^{N} \left(f_{n_{k+1}} - f_{n_k}\right) = f_{n_{N+1}}$, implying that $f_{n_k} \to f$ a.e. and in L^p. ☐

Corollary 18.4. If $f_n \to f$ in L^p, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e.

Proof. The proof of this is essentially the same as in the proof of Proposition 18.3. ☐

19 October 14, 2013

Convergence almost everywhere does not imply convergence in L^p.

Example 19.1. Take $f_n = \chi_{[n,n+1)}$. Certainly $f_n \to 0$ almost everywhere. However, $\|f_n\|_p = 1$ for every p, so that $f_n \not\to 0$ in L^p.

Also, in general, convergence in measure does not imply convergence in L^p.

Example 19.2. Take $f_n = n\chi_{[0,\frac{1}{n})}$. Then $f_n \to 0$ in measure, but $\|f_n\|_p \geq 1$ for $p \in [1, \infty]$, so that $f_n \not\to 0$ in L^p.

If $p = \infty$, then convergence in $L^p = L^\infty$ does not imply convergence in measure.

Example 19.3. Take $f_n = \frac{1}{n}$. Then $f_n \to 0$ in L^∞, but

$$\mu\left(\{x \in X \mid |f_n(x)| > \epsilon\}\right) = \infty$$

for every ϵ and $n \in \mathbb{N}$.

However, if $p \in [1, \infty)$, we do have that convergence in L^p implies convergence in measure.

Lemma 19.4 (Chebyshev's Inequality)**.** If f is integrable, then

$$\mu\left(\{x \in X \mid |f(x)| > \lambda\}\right) \leq \frac{1}{\lambda} \int_{\{|f| > \lambda\}} |f| \, d\mu \leq \frac{1}{\lambda} \|f\|_1 .$$

Proof. Let $S = \{x \in X \mid |f(x)| > \lambda\}$. Then $\lambda \leq |f|$ on S, so that

$$\lambda\mu(S) = \int_S \lambda \, d\mu = \int_S |f| \, d\mu \leq \int_X |f| \, d\mu \Rightarrow \mu(S) \leq \frac{1}{\lambda} \int_S \lambda \, d\mu \leq \frac{1}{\lambda} \|f\|_1 .$$

□

Proposition 19.5. If $f_n \to f$ in L^p, then $f_n \to f$ in measure.

Proof. Fix $\epsilon > 0$. Let

$$S_n = \{x \in X \mid |f_n(x) - f(x)| > \epsilon\} = \{x \in X \mid |f_n(x) - f(x)|^p > \epsilon^p\} .$$

Then by Chebyshev's inequality

$$\mu(S_n) < \frac{1}{\epsilon^p} \int_X |f_n - f|^p \, d\mu = \frac{1}{\epsilon^p} \|f_n - f\|_p^p .$$

Since $f_n \to f$ in L^p, taking $n \to 0$, it follows that $f_n \to f$ in measure. □

Proposition 19.6 (Uniform Integrability of One Function)**.** Say $f \in L^1$. Then for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $A \in \Sigma$ such that $\mu(A) < \delta$, $\int_A |f| \, d\mu < \epsilon$.

Proof. Fix $\epsilon > 0$. Let $S_\alpha = \{x \in X \mid |f(x)| > \alpha\}$. Let $f_\alpha = \chi_{S_\alpha}|f|$. Because $f \in L^1$, f is finite a.e., so that $f_\alpha \to 0$ a.e. as $\alpha \to \infty$. Further, $f_\alpha \leq |f|$, so that by Dominated Convergence,

$$\lim_{\alpha\to\infty} \int_{S_\alpha} |f|\, d\mu = \lim_{\alpha\to\infty} \int_X \chi_{S_\alpha}|f|\, d\mu = \int_X 0\, d\mu = 0.$$

Hence we can find $\alpha > 0$ such that $\int_{S_\alpha} |f|\, d\mu < \frac{\epsilon}{2}$. Choose $\delta = \frac{\epsilon}{2\alpha}$. Consider any $A \in \Sigma$ such that $\mu(A) < \delta$. Then

$$\begin{aligned}\int_A |f|\, d\mu &= \int_{A\cap S_\alpha^c} |f|\, d\mu + \int_{A\cap S_\alpha} |f|\, d\mu \\ &\leq \int_A |f|\, d\mu + \int_{S_\alpha} |f|\, d\mu \leq \int_A \alpha\, d\mu + \frac{\epsilon}{2} = \alpha\mu(A) + \frac{\epsilon}{2} < \alpha\delta + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

□

Note 19.7. The converse of Proposition 19.6 is not true. Take consider $f = 1$.

Definition 19.8 (Uniform Integrable)**.** A family of functions $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is *uniformly integrable* if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $A \in \Sigma$ such that $\mu(A) < \delta$, $\int_A |f_\alpha|\, d\mu < \epsilon$ for every $\alpha \in \mathcal{A}$.

Definition 19.9 (Tightness)**.** A family of functions $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is *tight* if for all $\epsilon > 0$, there exists $F_\epsilon \in \Sigma$ such that $\mu(F_\epsilon) < \infty$ and $\int_{F_\epsilon^c} |f_\alpha|\, d\mu < \epsilon$ for all $\alpha \in \mathcal{A}$.

Remark 19.10. A finite family of functions $\{f_1, \ldots, f_n\}$ in L^1 is uniformly integrable and tight. Further, if $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{g_\beta\}_{\beta\in\mathcal{B}}$ are uniformly integrable, then $\{f_\alpha\}_{\alpha\in\mathcal{A}} \cup \{g_\beta\}_{\beta\in\mathcal{B}}$ is uniformly integrable. A similar statement holds for tightness.

If $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is a family of functions such that $|f_\alpha| \leq F$ for some $F \in L^1$, then $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is uniformly integrable. A similar statement holds for tightness.

Theorem 19.11 (Vitali's Convergence Theorem)**.** Let $p \in [1, \infty)$. Let $f_n \in L^p$ for each n. Then $f_n \to f$ in L^p if and only if $f_n \to f$ in measure and $\{f_n^p\}$ is uniformly integrable and tight.

Proof. Suppose $f_n \to f$ in measure and $\{f_n^p\}$ is uniformly integrable and tight. Fix $\epsilon > 0$. By tightness, we can find $F_\epsilon \in \Sigma$ such that $\mu(F_\epsilon) < \infty$ and $\int_{F_\epsilon^c} |f|^p\, d\mu < \epsilon$ for each n. By uniformly integrability, we can find $\delta > 0$ such that for every $A \in \Sigma$ with $\mu(A) < \delta$, $\int_A |f_n|^p\, d\mu < \epsilon$ for each $n \in \mathbb{N}$.

Pick $\lambda = \frac{\epsilon}{\mu(F_\epsilon)}$. Let $S_n = \left\{x \in X \,\middle|\, |f_n(x) - f(x)| > \lambda^{\frac{1}{p}}\right\}$. Since $f_n \to f$ in measure, we can find $N \in \mathbb{N}$ such that for all $n \geq N$, $\mu\left(S\right) < \delta$.

By Jensen's inequality, $|f_n - f_m|^p \leq 2^{p-1}\left(|f_n|^p + |f_m|^p\right)$ for $n, m \in \mathbb{N}$. So

$$\int_{F_\epsilon^c} |f_n - f_m|^p\, d\mu \leq 2^{p-1}\left(\int_{F_\epsilon^c} |f_n|^p\, d\mu + \int_{F_\epsilon^c} |f_m|^p\, d\mu\right) < 2^p\epsilon. \tag{3}$$

Finally, we have for $n, m \geq N$,

$$\begin{aligned}
\int_{F_\epsilon} |f_n - f_m|^p \, d\mu &= \int_{F_\epsilon \cap S_n} |f_n - f_m|^p \, d\mu + \int_{F_\epsilon \cap S_n^c} |f_n - f_m|^p \, d\mu \\
&\leq \int_{S_n} |f_n - f_m|^p \, d\mu \\
&\qquad + 2^{p-1} \left(\int_{F_\epsilon \cap S_n^c} |f_n - f|^p \, d\mu + \int_{F_\epsilon \cap S_n^c} |f_m - f|^p \, d\mu \right) \\
&\leq \int_{S_n} |f_n - f_m|^p \, d\mu + 2^{p-1} \left(\int_{F_\epsilon \cap S_n^c} \lambda \, d\mu + \int_{F_\epsilon \cap S_n^c} \lambda \, d\mu \right) \\
&\leq 2^{p-1} \left(\int_{S_n} |f_n|^p \, d\mu + \int_{S_n} |f_m|^p \, d\mu \right) + 2^p \int_{F_\epsilon} \lambda \, d\mu \\
&\leq 2^p \epsilon + 2^p \lambda \mu(F_\epsilon) = 2^{p+1} \epsilon.
\end{aligned}$$

This with Equation 3 shows that $\int_X |f_n - f_m|^p \, d\mu \leq 3 \cdot 2^p \epsilon$. It follows that $\{f_n\}$ is a Cauchy sequence in L^p. Since L^p is complete, $\{f_n\}$ converges to some $g \in L^p$. So by Proposition 19.5, $f_n \to g$ in measure. Since $f_n \to f$ in measure as well, it is easy to check that $f = g$ a.e. Thus, $f_n \to f$ in L^p and $f \in L^p$.

Conversely, suppose $f_n \to f$ in L^p. By Proposition 19.5, $f_n \to f$ in measure.

Fix $\epsilon > 0$. Because L^p is complete, $f \in L^p$. So $f^p \in L^1$, and by Proposition 19.6, we can find $\delta > 0$ such that for every $A \in \Sigma$ with $\mu(A) < \delta$, $\int_A |f|^p \, d\mu < \frac{\epsilon}{2^p}$. We can find N such that for $n \geq N$, $\|f_n - f\|_p^p < \frac{\epsilon}{2^p}$. Hence, for $n \geq N$,

$$\begin{aligned}
\int_A |f_n|^p \, d\mu &\leq 2^{p-1} \left(\int_A |f|^p \, d\mu + \int_A |f_n - f|^p \, d\mu \right) \\
&\leq 2^{p-1} \left(\int_A |f|^p \, d\mu + \|f_n - f\|_p^p \right) < \epsilon.
\end{aligned}$$

This shows that $\{f_n^p\}_{n \geq N}$ is uniformly integrable. By Remark 19.10, it follows that $\{f_n^p\}_{n \in \mathbb{N}}$ is uniformly integrable.

Let $S_\delta = \{x \in X \mid |f(x)| < \delta\}$. By Chebyshev's inequality, for each $\delta > 0$ $\mu(S_\delta^c) \leq \frac{1}{\delta} \|f^p\|_1 < \infty$. Surely $\chi_{S_\delta} f^p \to 0$ pointwise and $|\chi_{S_\delta} f^p| \leq |f^p|$. Since $f^p \in L^1$, by Dominated Convergence, $\lim_{\delta \to 0} \int_{S_\delta} |f|^p \, d\mu = 0$. So we can choose $\delta > 0$ such that $\int_{S_\delta} |f|^p \, d\mu < \frac{\epsilon}{2^p}$. Hence, for $n \geq N$,

$$\begin{aligned}
\int_{S_\delta} |f_n|^p \, d\mu &\leq 2^{p-1} \left(\int_{S_\delta} |f|^p \, d\mu + \int_{S_\delta} |f_n - f|^p \, d\mu \right) \\
&\leq 2^{p-1} \left(\int_{S_\delta} |f|^p \, d\mu + \|f_n - f\|_p^p \right) < \epsilon.
\end{aligned}$$

This shows that $\{f_n^p\}_{n \geq N}$ is tight. By Remark 19.10, it follows that $\{f_n^p\}_{n \in \mathbb{N}}$ is tight. $\square$

20 October 16, 2013

Today, we attempt to find condition under which functions are uniformly integrable.

Lemma 20.1. Let f_n be measurable for $n \in \mathbb{N}$. Then the following are equivalent:

1.
$$\lim_{\lambda\to\infty} \sup_{n\in\mathbb{N}} \int_{\{|f_n|>\lambda\}} |f_n| \, d\mu = 0;$$

2. there exist an increasing super-linear $\phi : [0, \infty) \to [0, \infty)$ such that
$$\sup_{n\in\mathbb{N}} \int_X \phi \circ |f_n| \, d\mu < \infty.$$

Proof. Assume that condition 1 holds. Then we can find a $\lambda_1 > 0$ such that $\sup_{n\in\mathbb{N}} \int_{\{|f_n|>\lambda_1\}} |f_n| \, d\mu < \frac{1}{2}$. Recursively, given λ_k, we can find $\lambda_{k+1} > \lambda_k + 1$ such that $\sup_{n\in\mathbb{N}} \int_{\{|f_n|>\lambda_{k+1}\}} |f_n| \, d\mu < \frac{1}{2^{k+1}}$. Let $\phi(x) = \sum_{k=1}^{\infty} x\chi_{[0,\lambda_k]^c}(x)$. Note that

$$\int_X \phi \circ |f_n| \, d\mu = \int_X \sum_{k=1}^{\infty} |f_n| \cdot \left(\chi_{[0,\lambda_k]^c} \circ |f_n|\right) \, d\mu = \sum_{k=1}^{\infty} \int_{\{|f_n|>\lambda_k\}} |f_n| \, d\mu < 1.$$

Clearly ϕ is increasing. Further for each k, $\frac{\phi(x)}{x} \geq k$ for $x > \lambda_k$. By design, $\lambda_k \to \infty$ as $k \to \infty$. Thus, it follows that ϕ is super-linear.

Conversely, suppose condition 2 holds. Fix $\epsilon > 0$. Then we can find $\lambda > 0$ such that for $x > \lambda$, $x < \epsilon\phi(x)$. Hence, for each $n \in \mathbb{N}$,

$$\int_{\{|f_n>\lambda\}} |f_n| \, d\mu < \epsilon \int_{\{|f_n|>\lambda\}} \phi \circ |f_n| \, d\mu \leq \epsilon \int_X \phi \circ |f_n| \, d\mu \leq \epsilon \sup_{n\in\mathbb{N}} \int_X \phi \circ f_n \, d\mu.$$

Since $\sup_{n\in\mathbb{N}} \int_X \phi \circ f_n \, d\mu < \infty$, and $\sup_{n\in\mathbb{N}} \int_{\{|f_n|>\lambda\}} |f_n| \, d\mu$ is decreasing as λ increases, this implies condition 1. $\square$

Theorem 20.2 (Conditions for Uniform Integrability)**.** If either condition in Lemma 20.1 holds, then $\{f_n\}$ is uniformly integrable.

Proof. Without loss of generality, we may assume only the first condition holds. Fix $\epsilon > 0$. We can find $\lambda > 0$ such that $\sup_{n\in\mathbb{N}} \int_{\{|f_n|>\lambda\}} |f_n| \, d\mu < \frac{\epsilon}{2}$. Choose $\delta = \frac{\epsilon}{2\lambda}$. Pick any $A \in \Sigma$ with $\mu(A) < \delta$. Then, writing $S = \{|f_n| > \lambda\}$,

$$\begin{aligned}\int_A |f_n| \, d\mu &= \int_{A\cap S} |f_n| \, d\mu + \int_{A\cap S^c} |f_n| \, d\mu \\ &\leq \sup_{m\in\mathbb{N}} \int_S |f_m| \, d\mu + \int_{A\cap S^c} \lambda \, d\mu \leq \frac{\epsilon}{2} + \int_A \lambda \, d\mu = \frac{\epsilon}{2} + \lambda\mu(A) = \epsilon.\end{aligned}$$

$\square$

Remark 20.3. If $\{f_n\}$ is uniformly integrable and $\sup_{n\in\mathbb{N}} \int_X |f_n|\, d\mu < \infty$, then the conditions of Lemma 20.1 hold.

Corollary 20.4. If $\mu(X) < \infty$ and $f_n \to f$ in measure and $\sup_{n\in\mathbb{N}} \|f_n\|_p < \infty$ for $p > 1$, then $f_n \to f$ in L^1 (and in L^q for $q < p$).

Theorem 20.5 (Simple Functions are Dense in L^p)**.** Let $p \in [1, \infty)$. Simple functions are dense in L^p.

Proof. Choose any $f \in L^p$. By Proposition 11.4, we can find simple functions $\{s_n\}$ and $\{t_n\}$ such that $s_n \nearrow f^+$ and $t_n \nearrow f^-$. Let $r_n = s_n + t_n$. Then r_n is simple, $|r_n| \leq |f|$ and $r_n \to f$ pointwise.

Thus $|r_n - f|^p \to 0$ pointwise. Also, $|r_n - f|^p \leq (|r_n| + |f|)^p \leq 2^p |f|^p \in L^1$. Hence, by Dominated convergence, $\int_X |r_n - f|^p\, d\mu \to \int_X 0\, d\mu = 0$. Hence $r_n \to f$ in L^p. □

Theorem 20.6 (C_c is Dense in L^p)**.** Let $p \in [1, \infty)$. If X is a metric space and there exists compact sets $K_n \subseteq K_{n+1}$ for $n \in \mathbb{N}$ such that $\bigcup_{n=1}^{\infty} K_n = X$, then C_c is dense in L^p.

Proof. Suppose X is compact. Consider any $f \in L^p$. Fix $\epsilon > 0$. For brevity, let $\Delta_{f,g} = \{x \in X \mid f(x) \neq g(x)\}$ for any two function f and g. Since $f^p \in L^1$, we can find $\delta > 0$ such that for any $A \in \Sigma$ with $\mu(A) < \delta$, $\int_A |f|^p\, d\mu < \epsilon^p$.

By Lusin's Theorem, we can find a continuous g such that $\mu(\Delta_{f,g}) < \delta$ and $|g| \leq |f|$ a.e. [Why?]. Hence, since X is compact, $g \in C_c(X)$, and

$$\int_X |f-g|^p\, d\mu = \int_{\Delta_{f,g}} |f-g|^p\, d\mu \leq 2^p \int_{\Delta_{f,g}} |f|^p\, d\mu \leq 2^p \mu(\Delta_{f,g}) = 2^p \epsilon^p.$$

So $\|f - g\|_p \leq 2\epsilon$. Taking $\epsilon \to 0$, it follows that $C_c(X)$ is dense in L^p.

Now, consider the general case. Let $f_n = \chi_{K_n} f$. Certainly $f - f_n \to 0$ pointwise, and $|f - f_n|^p \leq 2|f|^p \in L^1$. So by Dominated convergence, $f_n \to f$ in L^p. Hence, we can find $n \in \mathbb{N}$ such that $\|f_n - f\|^p < \epsilon$. Restricting our attention to the compact space K_n, by what we showed just above, we can find a continuous function g supported on K_n such that $\|g - f_n\|_p < \epsilon$ [Why?]. So $g \in C_c(X)$ and by Minkowski's inequality, $\|g - f\|_p \leq \|g - f_n\|_p + \|f_n - f\|_p < 2\epsilon$. Taking $\epsilon \to 0$, it follows that $C_c(X)$ is dense in $L^p(X)$. □

21 October 21, 2013

Definition 21.1 (Signed Measure)**.** Let X be a set and Σ a σ-algebra of X. Then μ is a *signed measure* on (X, Σ) if

1. $\mu : \Sigma \to [-\infty, \infty]$ or $\mu : \Sigma \to (-\infty, \infty]$;
2. $\mu(\emptyset) = 0$;
3. for $A_i \in \Sigma$ disjoint, $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

Note 21.2. In the above definition, for disjoint $A_i \in \Sigma$, $\sum_{i=1}^{\infty} \mu(A_i)$ must be absolutely convergent in order for that condition to make sense.

Example 21.3. If μ_1 and μ_2 are two positive measures with at least one finite, then $\mu = \mu_1 - \mu_2$ is a signed measure.

Definition 21.4 (Positive and Negative Set)**.** Let μ be a signed measure. A set $P \in \Sigma$ is *positive* if for every $A \subseteq P$, $\mu(A) \geq 0$. A set $N \in \Sigma$ is *negative* if for every $A \subseteq N$, $\mu(A) \leq 0$.

Remark 21.5 (Monotonicity of Signed Measures)**.** Let μ be a signed measure. Let $P \in \Sigma$ be positive. Let $A \subseteq P$. Then $\mu(A) \subseteq \mu(P)$. Similarly, if $N \in \Sigma$ is negative and $B \subseteq N$, then $\mu(B) \geq \mu(N)$.

Proof. Suppose $\mu(A) > \mu(P)$. Then $\mu(P - A) = \mu(P) - \mu(A) < 0$, contradiction the fact that P is positive. Hence $\mu(A) \leq \mu(P)$. A similar argument shows that $\mu(B) \geq \mu(N)$. □

Definition 21.6 (Null Set)**.** Let μ be a signed measure. A set $M \in \Sigma$ is *null* if for every $A \subseteq P$, $\mu(A) = 0$.

Remark 21.7. If a set is positive and negative, then it is null.

Lemma 21.8. Let μ be a signed measure. Let $A \in \Sigma$ with $|\mu(A)| < \infty$. Then there exists $N \in \Sigma$ such that $N \subseteq A$, N is negative and $\mu(N) \leq \mu(A)$.

Proof. Choose $\delta_1 = \sup\{\mu(B) \mid B \subseteq A, B \in \Sigma\}$. Note that $\delta_1 \geq 0$ since $\emptyset \subseteq A$. Then we can find $B_1 \in \Sigma$ such that $B_1 \subseteq A$ and $\mu(B_1) \geq \min\left\{\frac{\delta}{2}, 1\right\}$. Then given $B_1, \dots, B_n$, define $\delta_{n+1} = \sup\left\{\mu(B) \mid B \in \Sigma, B \subseteq A - \bigcup_{i=1}^{n} B_i\right\}$. Then we can choose $B_{n+1} \in \Sigma$ such that $B_{n+1} \subseteq A$ and $\mu(B_{n+1}) \geq \min\left\{\frac{\delta_{n+1}}{2}, 1\right\}$.

Let $B = \bigcup_{i=1}^{\infty} B_n$, and let $N = A - B$. Note that $\mu(B) \geq 0$, since each $\delta_n \geq 0$. Since $N \cap B = \emptyset$, so that $\mu(N) + \mu(B) = \mu(A) \Rightarrow \mu(N) = \mu(A) - \mu(B) \leq \mu(A)$.

Note that, by choice, the B_n's are disjoint. Hence, $\mu(B) = \sum_{n=1}^{\infty} \mu(B_n) < \infty$ (since $|\mu(A)| < \infty$ and $B \subseteq A$). So $\sum_{n=1}^{\infty} \min\left\{\frac{\delta_n}{2}, 1\right\} < \infty$, which implies that $\delta_n \to 0$ as $n \to \infty$.

Consider any $C \subseteq N$. Then $C \subseteq A - B \subseteq A - \bigcup_{n=1}^{N} B_N$ for each $N \in \mathbb{N}$. This implies, by definition of B_{n+1}, that $\mu(C) \leq \delta_{n+1}$. Taking $\delta \to 0$, it follows that $\mu(C) \leq 0$. Thus, N is negative. □

Theorem 21.9 (Hahn Decomposition)**.** If μ is a signed measure, there exists $N, P \in \Sigma$ such that $N \cap P = \emptyset$, $N \cup P = X$, N is negative, and P is positive. Such a decomposition is unique up to null sets.

Proof. Assume without loss of generality that $\mu(X) > -\infty$.

Let $\alpha = \inf\{\mu(B) \,|\, B \in \Sigma\}$. Note $\alpha \leq 0$, since $\emptyset \in \Sigma$. So we can find sets $B_n \in \Sigma$ such that $\mu(B_n) \geq \infty$, $\mu(B_{n+1}) \leq \mu(B_n)$ and $\mu(B_n) \to \alpha$.

Lemma 21.8 implies that for each n, we can find negative $B'_n \subseteq B_n$ such that $\mu(B'_n) \leq \mu(B_n)$. Let $N = \bigcup_{n=1}^{\infty} B'_n$. Note that N is negative. Hence, by Remark 21.5, $\mu(N) \leq \mu(B'_n)$ for each n. Also, $\alpha \leq \mu(N)$ by the definition of α. Since $\mu(B_n) \to \alpha$, for every $\epsilon > 0$, we can find m such that $\mu(B_m) < \alpha + \epsilon$. Hence $\alpha \leq \mu(N) \leq \mu(B'_m) \leq \mu(B_m) \leq \alpha + \epsilon$. Taking $\epsilon \to 0$, $\mu(N) = \alpha$.

Let $P = X - N$. Let $A \subseteq P$. Lemma 21.8 implies we can find negative $A' \subseteq A$ such that $\mu(A') \leq \mu(A)$. Since $A' \cup N$ is negative, monotonicity of the absolute value of the measures hold in $A' \cup N$ (this is easy to see). Hence, if $\mu(A') < 0$, then $\mu(N \cup A') < \mu(N) = \alpha$, which contradicts the definition of α. So, $\mu(A') = 0$. This implies $\mu(A) \geq 0$. Therefore, P is positive.

Suppose (N', P') and (N, P) are two such decomposition. Since N' is negative, $N' \cap P$ is negative. However, since P is positive $N' \cap P$ is also positive. This implies that $N' \cap P$ is a null set, so that $N' - N = N' \cap P$ is a null set as well. Similarly, $N - N'$ is a null set. A similar argument shows that $P' - P$ and $P - P'$ are null sets as well. This shows that the decompositions are unique up to null sets. $\square$

Theorem 21.10 (Jordan Decomposition)**.** If μ is a signed measure, there exists positive measures μ^+ and μ^-, at least one of which is finite, such that

1. $\mu = \mu^+ - \mu^-$;
2. the Hahn decomposition (N, P) of μ is such that $\mu^+(N) = \mu^-(P) = 0$.

Such a decomposition (μ^+, μ^-) is unique.

Proof. Let $A \in \Sigma$. Define $\mu^+(A) = \mu(A \cap P)$ and $\mu^- = -\mu(A \cap N)$. Since P is positive and N is negative, μ^+ and μ^- are both positive measures. By additivity, $\mu = \mu^+ - \mu^-$. Furthermore, one of $\mu(P)$ and $\mu(N)$ must be finite, so that one of μ^+ and μ^- is finite. This shows the existence of such a decomposition.

Uniqueness follows from the fact that the Hahn decomposition of μ is unique up to null sets, and $\mu^+ = \mu^- = \nu^+ = \nu^-$ on μ-null sets. $\square$

22 October 23, 2013

Definition 22.1 (Absolutely Continuous Measure)**.** Let μ and ν be positive measures on (X, Σ). We say ν is *absolutely continuous with respect to* μ if for every $A \in \Sigma$ such that $\mu(A) = 0$, $\nu(A) = 0$. We write $\nu \ll \mu$.

Example 22.2. Let μ be a positive measure and $g \geq 0$. Define $\nu(A) = \int_A g \, d\mu$. Then ν is a positive measure and if $f \in L^1(X, \nu)$, then $\int_X f \, d\nu = \int_X fg \, d\mu$.

Theorem 22.3 (Radon-Nikodým)**.** If μ and ν are σ-finite positive measures and $\nu \ll \mu$, then there exists a unique measurable $g \geq 0$ such that $\nu(A) = \int_A g \, d\mu$ for every $A \in \Sigma$.

Proof. Suppose μ and ν are finite. Let $\mathcal{F} = \left\{ f \geq 0 \,\middle|\, \int_A f \, d\mu \leq \nu(A) \, \forall \, A \in \Sigma \right\}$. Since $0 \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. If $f_1, f_2 \in \mathcal{F}$, then $\max\{f_1, f_2\}$ is certainly measurable and positive. Then $\max\{f_1, f_2\} \in \mathcal{F}$, since

$$\begin{aligned} \int_A \max\{f_1, f_2\} \, d\mu &= \int_{A \cap \{f_1 \leq f_2\}} f_2 \, d\mu + \int_{A \cap \{f_1 \geq f_2\}} f_1 \, d\mu \\ &= \nu(A \cap \{f_1 \leq f_2\}) + \nu(A \cap \{f_1 \geq f_2\}) = \nu(A). \end{aligned}$$

Suppose $f_n \in \mathcal{F}$ with $f_n \leq f_{n+1}$. Since $\lim_{n\to\infty} f_n$ is measurable and positive, by Monotone Convergence, $\int_A \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int_A f_n \, d\mu \leq \nu(A)$.

Let $\alpha = \sup_{f \in \mathcal{F}} \int_X f \, d\mu$. Note that $0 \leq \alpha \leq \nu(X) < \infty$. We can choose $f_n \in \mathcal{F}$ such that $\lim_{n\to\infty} \int_X f_n \, d\mu = \alpha$. Then $g_n = \max\{f_1, \ldots, f_n\} \in \mathcal{F}$. Further, $g = \sup_{n \in \mathbb{N}} f_n = \lim_{n\to\infty} g_n \in \mathcal{F}$, by above.

Define $\nu_0(A) = \nu(A) - \int_A g \, d\mu$. Since $g \in \mathcal{F}$, ν_0 is a positive measure. Fix $\epsilon > 0$. Then $\nu_0 - \epsilon\mu$ is a signed measure. Let (P, N) be the Hahn decomposition of $\nu_0 - \epsilon\mu$. We would like to show that $(\nu_0 - \epsilon\mu)(P) = 0$. Let $h = \epsilon\chi_P + g$. Note that $g \leq h$ and

$$\begin{aligned} \int_A h \, d\mu &= \int_{A \cap P} \epsilon \, d\mu + \int_A g \, d\mu \leq \epsilon\mu(A \cap P) + \int_A g \, d\mu \\ &\leq \nu_0(A \cap P) + \int_A g \, d\mu \leq \nu_0(A) + \int_A g \, d\mu = \nu(A). \end{aligned}$$

So, $h \in \mathcal{F}$. Since $g \leq h$, $\int_X h \, d\mu = \alpha$, so that $\int_X (h - g) \, d\mu = 0$. This implies $h = g$ a.e. Therefore, $\chi_P = 0$ a.e., so that $\mu(P) = 0$. Therefore, since $\nu \ll \mu$, $\nu(P) = 0$. This implies that $\int_P g \, d\mu = 0$, so that $\nu_0(P) = 0$. Hence $(\nu_0 - \epsilon\mu)(P) = 0$. This implies that $\nu_0(A) \leq \epsilon\mu(A)$ for all $A \in \Sigma$. Taking $\epsilon \to 0$, it follows that $\nu_0(A) = 0$. As a result, $\nu(A) = \int_A g \, d\mu$.

Suppose there are g_1 and g_2 such that $\nu(A) = \int_A g_1 \, d\mu = \int_A g_2 \, d\mu$ for all $A \in \Sigma$. This automatically implies that $g_1 = g_2$ a.e.

Suppose now that μ and ν are σ-finite. Then $X = \bigcup_{n=1}^\infty A_n$ for some $A_n \in \Sigma$ with $\mu(A_n)$ and $\nu(A_n)$ finite. Without loss of generality, we may assume that $A_n \subseteq A_{n+1}$. Define $\nu_n(A) = \nu(A \cap A_n)$ and $\mu_n(A) = \mu(A \cap A_n)$. Certainly

$\nu_n \ll \mu_n$, so that there exists $g_n \geq 0$ such that $\nu_n(A) = \int_A g_n \, d\mu_n$ for all $A \in \Sigma$. By uniqueness of the g_n's, we must have $g_n = g_{n+1}\chi_{A_n}$. So $g_n \leq g_{n+1}$. Let $g = \lim_{n\to\infty} g_n$. Hence, by Monotone Convergence, for each A,

$$\nu(A) = \lim_{n\to\infty} \nu(A \cap A_n) = \lim_{n\to\infty} \nu_n(A) = \lim_{n\to\infty} \int_A g_n \, d\mu = \int_A g \, d\mu.$$

□

Definition 22.4 (Radon-Nikodým Derivative)**.** The function g in the statement of the theorem above is called the *Radon-Nikodým derivative of ν with respect to μ*, and is also denoted $g = \frac{d\nu}{d\mu}$.

Definition 22.5 (Total Variation)**.** The *total variation* of a signed measure μ is $|\mu|$, which is defined as $|\mu|(A) = \mu^+(A) + \mu^-(A)$ for all $A \in \Sigma$.

Definition 22.6 (Norm of Measure)**.** The norm of a signed measure μ is defined as $\|\mu\| = |\mu|(X)$.

23 October 25, 2013

Definition 23.1 (Singular Measure)**.** If μ and ν are positive measures on (X, Σ), we say μ and ν are *mutually singular* ($\mu \perp \nu$) if there exists $A, B \in \Sigma$ such that $A \cup B = X$, $A \cap B = \emptyset$, and $\mu(B) = \nu(A) = 0$.

Example 23.2. If μ is a signed measure, then $\mu^+ \perp \mu^-$, where (μ^+, μ^-) is the Jordan decomposition of μ.

Example 23.3. $\lambda \perp \delta_x$, where δ_x is the delta measure with mass at x.

Theorem 23.4 (Lebesgue Decomposition)**.** Let μ be a positive measure and ν be a σ-finite positive measure or a finite signed measure. Then there exists unique signed measures ν_{ac} and ν_{s} such that $\nu = \nu_{\mathrm{ac}} + \nu_{\mathrm{s}}$, $|\nu_{\mathrm{ac}}| \ll \mu$, and $|\nu_{\mathrm{s}}| \perp \mu$.

Proof. Suppose ν is a finite positive measure. Let $\mathcal{N} = \{A \in \Sigma \,|\, \mu(A) = 0\}$. Let $\alpha = \sup\{\nu(A) \,|\, A \in \mathcal{N}\}$. Note that $\alpha \le \nu(X) < \infty$. So we can find $A_i \in \mathcal{N}$ such that $\nu(A_i) \nearrow \alpha$ as $i \to \infty$.

Let $N = \bigcup_{i=1}^\infty A_i$. Note $\mu(N) = 0$ and $\nu(N) = \lim_{i\to\infty} \nu(A_i) = \alpha$. For $A \in \Sigma$, define $\nu_{\mathrm{ac}}(A) = \nu(A \cap N^c)$ and $\nu_{\mathrm{s}}(A) = \nu(A \cap N)$. Clearly $\nu = \nu_{\mathrm{ac}} + \nu_{\mathrm{s}}$.

Since $\nu_{\mathrm{s}}(N^c) = \nu(N^c \cap N) = 0$ and $\mu(N) = 0$, $\nu_{\mathrm{s}} \perp \mu$.

Consider any $A \in \Sigma$ with $\mu(A) = 0$. So $A \cup N \in \mathcal{N}$, so that

$$\alpha \ge \nu(A \cup N) = \nu(A \cap N^c) + \nu(N) = \nu(A \cap N^c) + \alpha.$$

This implies that $\nu_{\mathrm{ac}}(A) = \nu(A \cap N^c) = 0$. Hence $\nu_{\mathrm{ac}} \ll \mu$.

Now suppose ν is a σ-finite positive measure. We can find $B_n \subseteq B_{n+1}$ in Σ such that $\bigcup_{n=1}^\infty B_n = X$ and $\mu(B_n) < \infty$ for each n. For each n and $A \in \Sigma$, define $\nu^{(n)}(A) = \nu(A \cap B_n)$. Then $\nu^{(n)}$ is a positive finite measure, so by above, we can decompose $\nu^{(n)}$ as $\nu^{(n)} = \nu^{(n)}_{\mathrm{ac}} + \nu^{(n)}_{\mathrm{s}}$ such that $\nu^{(n)}_{\mathrm{ac}}$ and $\nu^{(n)}_{\mathrm{s}}$ are positive finite measures and $\nu^{(n)}_{\mathrm{ac}} \ll \mu$ and $\nu^{(n)}_{\mathrm{s}} \perp \mu$. So for each n, there exists N_n such that $\mu(N_n) = 0$ and $\nu^{(n)}_{\mathrm{s}}(N_n^c) = 0$. Let $N = \bigcup_{n=1}^\infty N_n$. Note $\mu(N) = 0$.

For $A \in \Sigma$, define $\nu_{\mathrm{ac}}(A) = \nu(A \cap N^c)$ and $\nu_{\mathrm{s}}(A) = \nu(A \cap N)$. Certainly, $\nu_{\mathrm{s}} \perp \mu$. Choose $A \in \Sigma$ such that $\mu(A) = 0$. Since $X = \bigcup_{n=1}^\infty B_n$ and $B_n \subseteq B_{n+1}$,

$$\begin{aligned}
\nu_{\mathrm{ac}}(A) = \nu(A \cap N^c) &= \lim_{n\to\infty} \nu(A \cap N^c \cap B_n) \\
&\le \lim_{n\to\infty} \nu(A \cap N_n^c \cap B_n) \\
&\le \lim_{n\to\infty} \nu^{(n)}(A \cap N_n^c) \le \lim_{n\to\infty} \nu^{(n)}_{\mathrm{ac}}(A) = 0
\end{aligned}$$

Hence, $\nu_{\mathrm{ac}} \ll \mu$.

Finally, suppose ν is a finite signed measure. Let (ν^+, ν^-) be the Jordan decomposition of ν. ν^+ and ν^- are finite positive measures, so that by above, we can find Lebesgue decompositions $\nu^+ = \nu^+_{\mathrm{ac}} + \nu^+_{\mathrm{s}}$ and $\nu^- = \nu^-_{\mathrm{ac}} + \nu^-_{\mathrm{s}}$ such that

$\nu_{\mathrm{ac}}^{+}, \nu_{\mathrm{ac}}^{-} \ll \mu$ and $\nu_{\mathrm{s}}^{+}, \nu_{\mathrm{s}}^{-} \perp \mu$. Choosing $\nu_{\mathrm{ac}} = \nu_{\mathrm{ac}}^{+} - \nu_{\mathrm{ac}}^{-}$ and $\nu_{\mathrm{s}} = \nu_{\mathrm{s}}^{+} - \nu_{\mathrm{s}}^{-}$ gives the desired decomposition of ν.

It remains to show uniqueness. Suppose there were another decomposition $\nu = \eta_{\mathrm{ac}} + \eta_{\mathrm{s}}$ such that $\eta_{\mathrm{ac}} \ll \mu$ and $\eta_{\mathrm{s}} \perp \mu$. Then $|\eta_{\mathrm{ac}} - \nu_{\mathrm{ac}}| \ll \mu$ and $|\eta_{\mathrm{s}} - \nu_{\mathrm{s}}| \perp \mu$. Further, since $\nu_{\mathrm{ac}} + \nu_{\mathrm{s}} = \eta_{\mathrm{ac}} + \eta_{\mathrm{s}}$, $\nu_{\mathrm{ac}} - \eta_{\mathrm{ac}} = \eta_{\mathrm{s}} - \mu_{\mathrm{s}}$. This implies that $|\eta_{\mathrm{s}} - \nu_{\mathrm{s}}| \ll \mu$ as well. Hence, this implies that $|\eta_{\mathrm{ac}} - \nu_{\mathrm{ac}}| = 0$, implying that $\eta_{\mathrm{ac}} = \nu_{\mathrm{ac}}$. This proves the uniqueness of such a decomposition. □

As an application of the Radon-Nikodým derivative, we will (eventually) show that the dual of L^p is L^q, where $\frac{1}{p} + \frac{1}{q} = 1$ and $p \in [1, \infty)$. The measure μ will be σ-finite.

Definition 23.5 (Bounded Operator)**.** Let X and Y be Banach spaces. Let $T : X \to Y$ be linear. Then T is *bounded* if there exists $C > 0$ such that $\|Tx\|_Y \leq C \|x\|_X$ for all $x \in X$, or $\sup_{x \in X - \{0\}} \frac{\|Tx\|_Y}{\|x\|_X} < \infty$.

Proposition 23.6 (Bounded Operator is Continuous)**.** Let X and Y be Banach spaces. Let $T : X \to Y$ be linear. Then T is continuous if and only if it is bounded.

Proof. Suppose T is bounded. Then there exists C such that $\|Tx\|_Y \leq C \|x\|_X$ for all $x \in X$. So for every $x_1, x_2 \in X$, $\|Tx_1 - Tx_2\|_Y = \|T(x_1 - x_2)\|_Y \leq C \|x_1 - x_2\|_X$. So T is Lipschitz, and hence continuous.

Conversely, suppose T is continuous. Then we can find $\delta > 0$ such that for $\|x\|_X < \delta$, $\|Tx\|_Y < 1$. Hence, for any $x \in X$, by linearity,

$$\left\| T \frac{\delta x}{2 \|x\|} \right\|_Y = \frac{\delta}{2 \|x\|_X} \|Tx\|_Y \leq 1 \Rightarrow \|Tx\|_Y \leq \frac{2}{\delta} \|x\|_X .$$

Hence T is bounded. □

Note 23.7. Note that the above proof actually also show that T is continuous if and only if it is Lipschitz.

24 October 28, 2013

Definition 24.1. Let X and Y be Banach spaces. Define

$$B(X,Y) = \{T : X \to Y \,|\, T \text{ is bounded and linear}\}.$$

For $T \in B(X,Y)$, define

$$\|T\| = \sup_{x \in X - \{0\}} \frac{\|Tx\|_Y}{\|x\|_X}.$$

Proposition 24.2. $B(X,Y)$ is a Banach space.

Proof. We must show that $B(X,Y)$ is a complete normed vector space. If $\|T\| = 0$, then $\|Tx\|_Y = 0$ for all $x \in X - \{0\}$, so that $T = 0$. If $\alpha > 0$, $\frac{\|\alpha Tx\|_Y}{\|x\|_X} = \alpha \frac{\|Tx\|_Y}{\|x\|_X}$, so that taking supremums, $\|\alpha T\| = \alpha \|T\|$. Finally, for any $S, T \in B(X,Y)$ and $x \in X$,

$$\|(S+T)(x)\|_Y \le \|Sx\|_Y + \|Tx\|_Y \le \|S\| \|x\|_X + \|T\| \|x\|_X = (\|S\| + \|T\|) \|x\|_X$$

which implies that $\|S+T\| = \sup_{x \neq 0} \frac{\|(S+T)(x)\|_Y}{\|x\|_X} \le \|S\| + \|T\|$. Hence, $B(X,Y)$ is a normed vector space.

Let $\{T_n\}$ be a Cauchy sequence of $B(X,Y)$. Consider any $x \in X$. Then $\{T_n(x)\}$ is a Cauchy sequence in Y, so that $T_n(x) \to T(x)$ for some $T(x) \in Y$. So for $\alpha, \beta \in \mathbb{R}$ and $x_1, x_2 \in X$,

$$T(\alpha x_1 + \beta x_2) \leftarrow T_n(\alpha x_1 + \beta x_2) = \alpha T_n(x_1) + \beta T_n(x_2) \to \alpha T(x_1) + \beta T(x_2).$$

Hence T is linear. Since T_n is Lipschitz, $T_n \to T$ uniformly, so that T is continuous, and thus bounded, and $\|T_n - T\| \to 0$. Hence $T \in B(X,Y)$, and $B(X,Y)$ is Cauchy complete. $\square$

Definition 24.3 (Dual of Space)**.** The *dual* of a Banach space X is the space $X^* = B(X, \mathbb{R})$.

Theorem 24.4 (Duality of L^p)**.** Let (X, Σ, μ) a σ-finite measure space. Let $p \in [1, \infty)$. Pick q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then there exists a bijective linear isometry between $(L^p)^*$ and L^q.

Proof. Let $g \in L^q$. Define $T_g : L^p \to \mathbb{R}$ by $T_g(f) = \int_X fg \, d\mu$. Certainly T_g is linear and by Hölder's inequality, $|T_g(f)| \le \|f\|_p \|g\|_q$, so that T_g is bounded. Define $\phi : L^q \to (L^p)^*$ by $\phi(g) = T_g$. Certainly ϕ is linear.

Hence $T_g \in (L^p)^*$. Also, by the Duality equality,

$$\|T_g\|_{(L^p)^*} = \sup_{f \in L^p - \{0\}} \frac{|T_g(f)|}{\|f\|_p} = \|g\|_q.$$

So ϕ is an isometry. If $g, h \in L^q$ and $\phi(g) = \phi(h)$, then $\phi(g-h) = 0$, so that $\|\phi(g-h)\|_{(L^p)^*} = 0$. Since ϕ is an isometry, $\|g - h\|_{L^q} = 0$, which implies $g = h$ in L^q. Hence ϕ is injective.

It remains to show that ϕ is surjective. Suppose $\mu(X) < 0$. Pick any $T \in (L^p)^*$. We need to show that there exists a $g \in L^q$ such that $T = \phi(g)$. For $A \in \Sigma$, $\mu(A) \le \mu(X) < \infty$, so that $\chi_A \in L^p$. Define $\nu(A) = T(\chi_A)$.

Certainly $\nu(\emptyset) = T(\chi_\emptyset) = T(0) = 0$. Suppose $A_i \in \Sigma$ for $i \in \mathbb{N}$ are disjoint. Let $B_n = \bigcup_{i=1}^n A_i$, and let $A = \bigcup_{i=1}^\infty A_i$. We have

$$\nu(B_n) = T(\chi_{B_n}) = \sum_{i=1}^n T(\chi_{A_i}) = \sum_{i=1}^n \mu(A_i).$$

Note that $0 \le \chi_{B_n} \le \chi_{B_{n+1}} \le \chi_A \in L^p$ and $\chi_{B_n}^p \to \chi_A^p$ pointwise. Then by Dominated Convergence,

$$\begin{aligned}\sum_{i=1}^\infty \nu(A_i) = \lim_{n\to\infty} \sum_{i=1}^n \nu(A_i) = \lim_{n\to\infty} \nu(B_n) &= \lim_{n\to\infty} \int_X \chi_{B_n}\, d\nu \\ &= \int_X \chi_A \, d\nu = \nu(A)\end{aligned}$$

Hence ν is a finite signed measure. Further, if $A \in \Sigma$ such that $\mu(A) = 0$, then $\chi_A = 0$ in L^p, so that $T(\chi_A) = 0$, so that $\nu(A) = 0$. Hence, $\nu \ll \mu$.

So by Radon-Nikodým, there exists $g \in L^1$ such that $\nu(A) = \int_A g\, d\mu$. It suffices to show that $g \in L^q$ and $T(f) = \int_X fg\, d\mu$ for all $f \in L^p$.

We already have $T\chi_A = \int_X \chi_A g\, d\mu = \int_A g\, d\mu$ for all $A \in \Sigma$. Hence, by linearity, $Ts = \int_X sg\, d\mu$ for all simple functions s. Thus, by Corollary 17.6,

$$\|g\|_{L^q} = \sup_{\substack{s \in L^q - \{0\} \\ s \text{ simple}}} \frac{1}{\|s\|_{L^p}} \left| \int_X sg\, d\mu \right| = \sup_{\substack{s \in L^q - \{0\} \\ s \text{ simple}}} \frac{|Ts|}{\|s\|_{L^p}} \le \|T\| < \infty.$$

So $g \in L^q$.

Now consider any $f \in L^p$. Fix $\epsilon > 0$. By density of simple functions in L^p, we can find simple s such that $|s| \le |f|$ and $\|s - f\|_p < \epsilon$. Since T is continuous, and thus bounded, $|T(s - f)| \le C \|s - f\|_p < C\epsilon$ for some constant $C > 0$. So,

$$\begin{aligned}\left| T(f) - \int_X fg\, d\mu \right| &\le |T(f-s)| + \left| T(s) - \int_X sg\, d\mu \right| + \left| \int_X (s-f) g\, d\mu \right| \\ &< C\epsilon + 0 + \|s - f\|_p \|g\|_q < (C + \|g\|_q)\epsilon.\end{aligned}$$

Taking $\epsilon \to 0$, it follows that $T(f) = \int_X fg\, d\mu$, and $T = \phi(g)$. Hence ϕ is surjective, and thus is a bijective linear isometry. $\square$

25 October 30, 2013

Today's goal is to introduce product measures and determine when iterated integrals are defined and can be switched. Let (X, Σ, μ) and (Y, τ, ν) be σ-finite positive measures.

Definition 25.1 (Product σ-algebra)**.** Let $\Sigma \times \tau = \{A \times B \mid A \in \Sigma,\, B \in \tau\}$. Let $\Sigma \otimes \tau = \sigma(\Sigma \times \tau)$ be the *product σ-algebra* of Σ and τ.

Lemma 25.2. Say $C \in \Sigma \otimes \tau$. Then for every $x \in X$ and $y \in Y$, define

$$S_x(C) = \{y \in Y \mid (x, y) \in C\} \qquad \text{and} \qquad T_y(C) = \{x \in X \mid (x, y) \in C\}.$$

Then for every $x \in X$ and $y \in Y$, $S_x(C) \in \tau$ and $T_y(C) \in \Sigma$.

Proof. Let

$$\mathcal{C} = \{C \in \Sigma \otimes \tau \mid \forall\, x \in X \text{ and } y \in Y,\, S_x(C) \in \tau \text{ and } T_y(C) \in \Sigma\}.$$

Consider any $A \in \Sigma$ and $B \in \tau$. If $x \in A$, then $S_x(A \times B) = B \in \tau$, and if $x \notin A$, then $S_x(A \times B) = \emptyset \in \tau$. Similarly, if $y \in B$, then $T_y(A \times B) = A \in \Sigma$, and if $y \notin B$, then $T_y(A \times B) = \emptyset \in \Sigma$. Hence, $A \times B \in \mathcal{C}$. It follows that $\mathcal{C} \supseteq \Sigma \times \tau$.

Certainly $\emptyset, X \times Y \in \mathcal{C}$. Consider any $C \in \mathcal{C}$. Then for any $x \in X$, $S_x(C^c) = (S_x(C))^c \in \tau$. Similarly, for any $y \in Y$, $T_y(C^c) = (T_y(C))^c \in \Sigma$. Hence $C^c \in \mathcal{C}$.

Suppose $C_n \in \mathcal{C}$ for $n \in \mathbb{N}$. Let $C = \bigcup_{n=1}^{\infty} C_n$. For any $x \in X$, $S_x(C) = \bigcup_{n=1}^{\infty} S_x(C_n) \in \tau$. For any $y \in Y$, $T_y(C) = \bigcup_{n=1}^{\infty} T_y(C_n) \in \Sigma$. So, $\bigcup_{n=1}^{\infty} C_n \in \mathcal{C}$.

So $\mathcal{C}$ is a σ-algebra. Since $\mathcal{C} \supseteq \Sigma \times \tau$, it follows that $\mathcal{C} \supseteq \Sigma \otimes \tau$. So, $\mathcal{C} = \Sigma \otimes \tau$. $\square$

Note 25.3. The converse of the above lemma is false.

Lemma 25.4. Let $f_C(x) = \nu(S_x(C))$ and $g_C(y) = \mu(T_y(C))$. Then for every $C \in \Sigma \otimes \tau$, f_C is Σ-measurable and g_C is τ-measurable.

Proof. Suppose $\mu(X)$ and $\nu(X)$ are finite. Let

$$\Lambda = \{C \in \Sigma \otimes \tau \mid f_C \text{ is } \Sigma\text{-measurable and } g_C \text{ is } \tau\text{-measurable}\}.$$

Let $A \in \Sigma$ and $B \in \tau$. Then $f_C = \nu(B)\chi_A$ and $g_C = \mu(A)\chi_B$. Certainly f_C is Σ-measurable and g_C is τ-measurable. Hence $\Lambda \supseteq \Sigma \times \tau$.

Suppose $C_n \in \Lambda$ such that $C_n \subseteq C_{n+1}$. Let $C = \bigcup_{n=1}^{\infty} C_n$. Then note that $f_C = \sup_{n \in \mathbb{N}} f_{C_n}$. Since each of the f_{C_n} is Σ-measurable, f_C is Σ-measurable as well. A similar argument shows that g_C is τ-measurable. So $C \in \Lambda$.

Suppose $C, D \in \Lambda$ such that $C \subseteq D$. Then, since $\nu(X) < \infty$, $f_{D-C} = f_D - f_C$. Since f_D and f_C are Σ-measurable, f_{D-C} is Σ-measurable. A similar argument shows that g_{D-C} is τ-measurable. So $D - C \in \Lambda$.

Hence Λ is a λ-system. Since $\Lambda \supseteq \Sigma \times \tau$, which is a π-system, $\Lambda \supseteq \Sigma \otimes \tau$. Hence $\Lambda = \Sigma \otimes \tau$.

Now, suppose μ and ν are σ-finite. Then we can find $F_n \subseteq F_{n+1} \subseteq X$ and $G_n \subseteq G_{n+1} \subseteq Y$ such that $\bigcup_{n=1}^{\infty} F_n = X$ and $\bigcup_{n=1}^{\infty} G_n = Y$ and $\mu(F_n) < \infty$ and $\mu(G_n) < \infty$. Consider any $C \in \Sigma \otimes \tau$. Note that $f_C = \lim_{n\to\infty} f_{C\cap(F_n \times G_n)}$. Let $\mu_n(A) = \mu(A \cap F_n)$ and $\nu_n(B) = \nu(B \cap G_n)$ for each $A \in \Sigma$, $B \in \tau$ and $n \in \mathbb{N}$. Then what we showed above implies that $f_{C\cap(F_n\times G_n)}$ is Σ-measurable. Hence. f_C is Σ-measurable. A similar argument shows that g_C is τ-measurable. □

Proposition 25.5 (Uniqueness of Product Measure)**.** Let (X, Σ, μ) and (Y, τ, ν) be positive σ-finite measure spaces. Then there exists a unique measure π on $(X \times Y, \Sigma \otimes \tau)$ such that $\pi(A \times B) = \mu(A)\nu(B)$ for all $A \in \Sigma$ and $B \in \tau$.

Proof. We first show uniqueness. We know that two finite measures that agree on a π-system agrees on the σ-algebra generated by the π-system. Hence, by taking limits of μ and ν restricted to subspaces of finite measure, it follows that if two σ-finite measures agree on a π-system, then they agree on the σ-algebra generated by the π-system. Therefore, since $\Sigma \times \tau$ is a π-system, if two σ-finite measures agree on $\Sigma \times \tau$, then they agree on $\sigma(\Sigma \times \tau) = \Sigma \otimes \tau$. Hence, they agree everywhere. This proves uniqueness of such a product measure.

We know show the existence of such a product measure. By Lemma 25.4 f_C and g_C are measurable. Let $\pi_1(C) = \int_X f_C \, d\mu$. So $\pi_1(\emptyset) = \int_X f_\emptyset \, d\mu = \int_X 0 \, d\mu = 0$. Further, if $C_n \in \Sigma \otimes \tau$ are disjoint, then

$$\pi_1\left(\bigcup_{n=1}^{\infty} C_n\right) = \lim_{N\to\infty} \pi_1\left(\bigcup_{n=1}^{N} C_n\right) = \lim_{N\to\infty} \sum_{n=1}^{N} \pi_1(C_n) = \sum_{n=1}^{\infty} \pi_1(C_n).$$

Hence, π_1 is a measure on X. Further, if $A \in \Sigma$ and $B \in \tau$, then

$$\pi_1(A \times B) = \int_X \nu(B)\chi_A \, d\mu = \mu(A)\nu(B).$$

This shows the existence of such a product measure. □

Note 25.6. By defining $\pi_2(C) = \int_Y g_C \, d\nu$, the same argument as in the proof above shows that π_2 is a measure and $\pi_2(A \times b) = \mu(A)\nu(B)$ for $A \in \Sigma$ and $B \in \tau$. Moreover, uniqueness implies $\pi_1 = \pi_2$, which implies

$$\int_X \int_Y \chi_C(x,y) \, d\nu(y) \, d\mu(x) = \int_Y \int_X \chi_C(x,y) \, d\mu(x) \, d\nu(y),$$

which gives states that one may change the order of integrals for characteristic functions! As one might suspect, this can be generalized, under certain condition, other measurable and/or integrable functions.

26 November 1, 2013

Definition 26.1. Let $f : X \times Y \to [-\infty, \infty]$ be measurable. Define $Sf_x(y) = f(x,y)$ and $Tf_y(x) = f(x,y)$.

Lemma 26.2. Let $f : X \times Y \to \mathbb{R}$ be measurable. Then Sf_x is τ-measurable and Tf_y is σ-measurable for all $x \in X$ and $y \in Y$.

Proof. For any open $U \subseteq \mathbb{R}$, $Sf_x^{-1}(U) = S_x(f^{-1}(U))$. Since $f^{-1}(U)$ is $\Sigma \otimes \tau$-measurable, by Lemma 25.2, $S_x(f^{-1}(U))$ is τ-measurable. Hence Sf_x is τ-measurable. A similar argument shows that Tf_y is Σ-measurable. $\square$

Theorem 26.3 (Tonell's Theorem)**.** Suppose $f : X \times Y \to [0, \infty]$. Then let $F(x) = \int_Y Sf_x \, d\nu$ and $G(y) = \int_X Tf_y \, d\mu$. Then F is Σ-measurable and G is τ-measurable, and

$$\int_X F \, d\mu = \int_Y G \, d\nu = \int_{X \times Y} f \, d\pi.$$

Proof. Suppose $f = \chi_C$ for some $C \in \Sigma \otimes \tau$. From Lemma 25.4 and Note 25.6, $F = \nu(S_x(C))$ and is τ-measurable, and

$$\int_{X \times Y} \chi_C \, d\pi = \int_X f_C \, d\mu = \int_X \nu(S_x(C)) \, d\mu(x) = \int_X F \, d\mu.$$

Hence, by linearity, F is τ-measurable and $\int_{X \times Y} f \, d\pi = \int_X F \, d\mu$ for non-negative simple functions f.

We know we can find non-negative simple functions $s_n \nearrow f$ pointwise. For each n, define $S_n(x) = \int_X s_n(x,y) \, d\nu(y)$. Then by Monotone Convergence, $S_n(x) \to F(x)$ for each $x \in X$. Note that that S_n is simple for each n. So each S_n is τ-measurable, by what we showed above. Hence, F is also τ-measurable.

Further, since $s_n \le s_{n+1}$, it follows that $S_n \le S_{n+1}$, so that, again, by Monotone Convergence, $\int_X S_n \, d\mu \to \int_X F \, d\mu$. By above, $\int_{X \times Y} s_n \, d\pi = \int_X S_n \, d\mu$ for each n, so that, again, by Monotone Convergence,

$$\int_{X \times Y} f \, d\pi = \lim_{n \to \infty} \int_{X \times Y} s_n \, d\pi = \lim_{n \to \infty} \int_X S_n \, d\mu = \int_X F \, d\mu.$$

A similar argument shows G is Σ-measurable and $\int_{X \times Y} f \, d\pi = \int_Y G \, d\nu$. $\square$

Theorem 26.4 (Fubini's Theorem)**.** Say $f \in L^1$. Suppose $\int_{X \times Y} f^+ \, d\pi < \infty$ or $\int_{X \times Y} f^- \, d\pi < \infty$. Let $F(x) = \int_Y Sf_x \, d\nu$ and $G(y) = \int_X Tf_y \, d\mu$. Then $F(x)$ is defined μ-a.e. and $G(y)$ is defined ν-a.e., F is Σ-measurable and G is τ-measurable, and

$$\int_X F \, d\mu = \int_Y G \, d\nu = \int_{X \times Y} f \, d\pi.$$

Proof. Without loss of generality, assume $\int_{X\times Y} f^- \, d\pi < \infty$. Write $f = f^+ - f^-$. Define $F^+(x) = \int_Y S(f_x^+) \, d\nu$, $F^-(x) = \int_Y S(f_x^-) \, d\nu$, $G^+(x) = \int_X T(f_y^+) \, d\mu$, and $G^-(x) = \int_X T(f_y^-) \, d\mu$. Then by Tonelli's Theorem, F^+ and F^- are τ-measurable, and G^+ and G^- are Σ-measurable, and $\int_{X\times Y} f^+ \, d\pi = \int_X F^+ \, d\mu = \int_Y G^+ \, d\nu$ and $\int_{X\times Y} f^- \, d\pi = \int_X F^- \, d\mu = \int_Y G^- \, d\nu$

Since $\int_{X\times Y} f^- \, d\pi = \int_X F^- \, d\mu < \infty$, F^- is finite μ-a.e. Similarly, G^- is ν-finite a.e. Note that $S(f_x^+) + S(f_x^-) = Sf_x$. So $F = F^+ - F^-$, and F is defined μ-a.e. Similarly, G is defined μ-a.e. Hence, since F^+ and F^- are Σ-measurable, F is Σ-measurable. Similarly, G is τ-measurable. So, by linearity,

$$\int_{X\times Y} f \, d\pi = \int_{X\times Y} f^+ \, d\pi + \int_{X\times Y} f^- \, d\pi = \int_X F^+ \, d\mu + \int_X F^- \, d\mu = \int_X F \, d\mu.$$

Similarly,

$$\int_{X\times Y} f \, d\pi = \int_Y G \, d\nu.$$

□

Note 26.5. If $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is π-measurable and it's integral exists and Fubini's Theorem holds, it need not be that $F(x)$ exist everywhere. Take

$$f(x, y) = \begin{cases} 0 & x \neq 0 \text{ or } y = 0 \\ \frac{1}{y} & x = 0 \text{ and } y \neq 0 \end{cases}.$$

Then $\int_{\mathbb{R}} f(0, y) \, dy$ does not exist.

Remark 26.6. Even if the iterated integrals exist and are finite, the product integral may not exist and the interated integrals may not be equal. This was done in one of the homeworks.

27 November 4, 2013

We go over a few applications of iterated integrals and introduce convolutions.

Proposition 27.1. Let X be σ-finite and $f : X \to [0, \infty]$ be measurable. Then

$$\int_X f\, d\mu = \int_{[0,\infty)} \mu(\{x \in X \,|\, f(x) \geq \alpha\}\, d\lambda(\alpha).$$

Proof. Consider the product space $X \times [0, \infty)$ with the measure $\mu \times \lambda$. Let $C = \{(x, y) \in X \times Y \,|\, y \leq f(x)\}$. Remember $y \geq 0$. Then note that

$$\pi(C) = \int_X \lambda(S_x(C))\, d\mu(x) = \int_X f(x)\, d\mu(x)$$

and

$$\pi(C) = \int_Y \mu(T_y(C))\, d\lambda(y) = \int_Y \mu(\{y \,|\, y \leq f(x)\})\, d\lambda(y).$$

By Tonelli's theorem, we are done. □

Corollary 27.2. Say $\phi : [0, \infty) \to [0, \infty)$ is increasing, bijective, and C^1. Then

$$\int_X \phi \circ f\, d\mu = \int_{[0,\infty)} \phi'(y) \cdot \mu(\{x \in X \,|\, f(x) \geq y\})\, d\lambda(y).$$

Proof. We already know that

$$\begin{aligned}
\int_X \phi \circ f\, d\mu &= \int_{[0,\infty)} \mu(\{x \in X \,|\, \phi \circ f(x) \geq y\})\, d\lambda(y) \\
&= \int_{[0,\infty)} \phi'(z) \cdot \mu(\{x \in X \,|\, \phi \circ f(x) \geq \phi(z)\})\, d\lambda(z) \\
&= \int_{[0,\infty)} \phi'(z) \cdot \mu(\{x \in X \,|\, f(x) \geq z\})\, d\lambda(z),
\end{aligned}$$

where the second inequality comes from change of variables and the thirds comes from the fact ϕ is increasing and bijective. □

Definition 27.3 (Convolution)**.** Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be measurable. The *convolution* of f and g is

$$f * g = \int_{\mathbb{R}^d} f(x - y)g(y)\, dy = \int_{\mathbb{R}^d} f(y)g(x - y)dy.$$

Claim 27.4. If $f, g \in L^1$, then $f * g$ is defined and is in L^1.

Proof.

$$\begin{aligned}
\int_{\mathbb{R}^d} |f * g|\mu &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y)g(y)\, dy \right| dx \\
&\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)||g(y)|\, dy\, dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)||g(y)|\, dx\, dy = \int_{\mathbb{R}^d} \|f\|_1 \, |g(y)|\, dy = \|f\|_1 \, \|g\|_1 .
\end{aligned}$$

Hence, $\|f * g\|_1 \leq \|f\|_1 \, \|g\|_1 < \infty$, and so $f * g \in L^1$. $\square$

28 November 6, 2013

Theorem 28.1. Let f, g be measurable. Then $\|f * g\|_r \leq \|f\|_p \|g\|_q$, where $r, p, q \in [1, \infty)$ and $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$.

Proof. Let p', q' and r' be the Hölder conjugates of p, q and r, respectively. By the Duality equality, it is enough to show that for all $h \in L^{r'}$,

$$\int_{\mathbb{R}^d} f * g(x)h(x)\, dx \leq \|h\|_{r'} \|f\|_p \|g\|_q .$$

Without loss of generality, assume $f, g, h \geq 0$. Then $f * g \geq 0$. Let μ be the product measure $\lambda \times \lambda$ on $\mathbb{R}^d \times \mathbb{R}^d$. For brevity, write $X = \mathbb{R}^d \times \mathbb{R}^d$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^d} f * g(x)h(x)\, dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y)f(y)h(x)\, dy\, dx \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x-y)^{\frac{p}{r}} g(y)^{\frac{q}{r}} \cdot f(x-y)^{\frac{p}{q'}} h(x)^{\frac{r'}{q'}} \cdot g(y)^{\frac{q}{p'}} h(x)^{\frac{r'}{p'}}\, d\lambda \\
&\leq \left(\int_X f(x-y)^p g(y)^q \, d\lambda \right)^{\frac{1}{r}} \left(\int_X f(x-y)^p h(x)^{r'}\, d\lambda \right)^{\frac{1}{q'}} \\
&\qquad \times \left(\int_X g(y)^q h(x) r' \, d\lambda \right)^{\frac{1}{p'}} \\
&\leq \|f\|_p^{\frac{p}{r}} \|g\|_q^{\frac{q}{r}} \cdot \|f\|_p^{\frac{p}{q'}} \|h\|_{r'}^{\frac{r'}{q'}} \cdot \|g\|_q^{\frac{q}{p'}} \|h\|_{r'}^{\frac{r'}{p'}} = \|f\|_p \|g\|_q \|h\|_r .
\end{aligned}$$

□

Definition 28.2 (Approximate Identities)**.** Let $\phi_n : \mathbb{R}^d \to [0, \infty]$ for each n. We say $\{\phi_n\}$ is an *approximate identity* if

1. for every n, $\int_{\mathbb{R}^d} \phi_n \, d = 1$;
2. for every $\delta > 0$, $\int_{\{|y| > \delta\}} \phi_n \, d \to 0$ as $n \to \infty$.

Example 28.3. An example using balls is:

$$\phi_n = \frac{\chi_{B\left(0, \frac{1}{n}\right)}}{\lambda\left(B\left(0, \frac{1}{n}\right)\right)} = \frac{\chi_{B\left(0, \frac{1}{n}\right)} n^d}{\lambda\left(B\left(0, 1\right)\right)}$$

Example 28.4. Another example is one where we scale a function. Let $\phi \geq 0$, $\phi \in L^1$, and $\int_{\mathbb{R}^d} \phi \, d\lambda = 1$. Then for every $\epsilon > 0$, choose

$$\phi_\epsilon = \frac{\phi\left(\frac{x}{\epsilon}\right)}{\epsilon^d}.$$

Definition 28.5. Say $f : \mathbb{R}^d \to [-\infty, \infty]$. Define $\tau_y f(x) = f(x - y)$.

Remark 28.6. If $f \in L^p$, then $\tau_y f \to f$ in L^p as $|y| \to 0$.

Proof. This was proven in Homework 8. $\square$

Remark 28.7 (General Minkowski's Inequality)**.** Let (X, Σ, μ), (Y, τ, ν) be two σ-finite measure spaces, $p \in [1, \infty]$, and $f : X \times Y \to \mathbb{R}$ is $(\Sigma \otimes \tau)$-measurable. Let $F(x) = \int_Y f(x, y)\, d\nu(y)$. Then

$$\|F\|_{L^p(X)} \leq \int_Y \|Tf_y\|_{L^p(X)}\, d\nu(y).$$

Proposition 28.8. Let $f \in L^p$ for $p \in [1, \infty)$ and let $\{\phi_n\}$ be an approximate identity. Then $f * \phi_n \to f$ in L^p.

Proof. Pick $0 < \epsilon < 1$. Then we can find δ such that for $|y| < \delta$, $\|\tau_y f - f\|_p < \epsilon$. We have for $x \in \mathbb{R}^d$,

$$\begin{aligned} f * \phi_n(x) - f(x) &= \int_{\mathbb{R}^d} f(x - y)\phi_n(y)\, dy - f(x) \\ &= \int_{\mathbb{R}^d} (\tau_y f - f)(x)\phi_n(y)\, dy. \end{aligned}$$

So by the general Minkowski's inequality,

$$\begin{aligned} \|f * \phi_n - f\|_p &\leq \int_{\mathbb{R}^d} \|\tau_y - f\|_p\, \phi_n(y)\, dy \\ &= \int_{\{|y|<\delta\}} \|\tau_y - f\|_p\, \phi_n(y)\, dy + \int_{\{|y|\geq\delta\}} \|\tau_y - f\|_p\, \phi_n(y)\, dy \\ &\leq \int_{\{|y|<\delta\}} \epsilon\phi_n(y)\, dy + \int_{\{|y|\geq\delta\}} 2\, \|f\|_p\, \phi_n(y)\, dy \\ &= \epsilon + 2\, \|f\|_p \int_{\{|y|\geq\delta\}} \phi_n(y)\, dy \end{aligned}$$

Taking $n \to \infty$, and then $\epsilon \to 0$, the result follows. $\square$

Remark 28.9. The previous proposition is false for $p = \infty$. Take $d = 1$ and $\phi_n = 2n\chi_{[-\frac{1}{n}, \frac{1}{n}]}$ and $f = \chi_A$, where $A = \bigcup_{n\in\mathbb{Z}}[2n, 2n + 1]$.

Proposition 28.10. Say f is measurable and $\int_{B(0,n)} |f|\, dx < \infty$ for all n. Say $\phi \in C_c^\infty$. Then $f * \phi \in C^\infty$.

Proof. This was shown in Homework 11. $\square$

29 November 7, 2013

We will begin talking about Fourier series. The motivation for studying Fourier series is the desire to express a function f as a series of sine and cosines:

$$f = \sum_{n\in\mathbb{N}} c_n e^{inx}.$$

Definition 29.1. For $p \in [1, \infty)$, define the space of L^p periodic functions as

$$L^p_{\text{per}} = \left\{ f : \mathbb{R} \to \mathbb{C} \,\middle|\, f(x+1) = f(x) \text{ a.e., and } \int_0^1 |f|^p dx < \infty \right\}$$

and define the norm of $f \in L^p_{\text{per}}$ to be

$$\|f\|_{L^p_{\text{per}}} = \int_0^1 |f|^p dx.$$

For $f, g \in L^2_{\text{per}}$, define their *inner product* to be

$$\langle f, g\rangle = \int_0^1 fg\, dx.$$

Definition 29.2. Define $e_n(x) = e^{2\pi inx} \in L^2_{\text{per}}$.

Remark 29.3. Note that for $n, m \in \mathbb{Z}$, $\langle e_m, e_n\rangle = \delta_{mn}$, where δ_{mn} is the Kronecker delta. We eventually would like to express f as $\sum_{n\in\mathbb{Z}} c_n e_n$, where $c_n \in \mathbb{C}$. Thus, to compute these coefficients, one might guess $c_n = \langle f, e_n\rangle$.

Definition 29.4 (Fourier Series Coefficient)**.** Given $f \in L^2_{\text{per}}$, for $n \in \mathbb{N}$, define the nth Fourier coefficient of f to be

$$\mathcal{F}f(n) = \int_0^1 f\overline{e_n}, dx = \int_0^1 fe^{-2\pi inx}, dx.$$

Definition 29.5. Let $S_N f = \sum_{n=-N}^{N} \mathcal{F}f(n)e_n$.

Our goal is to show that $S_N f$ converges to f in some sense as $N \to \infty$.

Lemma 29.6. Let $P_N = \text{span}\{e_{-N}, \dots, e_N\}$. Then $f - S_N f$ is orthogonal to P_N. In particular, $f - S_N f$ is orthogonal to $S_N f$.

Proof. It suffices to show that $\langle f - S_N f, e_n\rangle = 0$ for all $n \in \{-N, \dots, N\}$. Pick any such n. Then

$$\langle f, e_n\rangle = \mathcal{F}f(n) = \langle S_N f, e_n\rangle.$$

Hence $\langle f - S_N f, e_n\rangle = 0$. □

Corollary 29.7. $\|S_N f - f\|_{L^2_{\text{per}}} \le \|p_N - f\|_{L^2_{\text{per}}}$ for all $p_N \in P_N$.

Proof. We can write $p_N - f = S_N f - f + p_N - S_N f$. Since $S_N - f$ is orthogonal to $p_N - f$ and $p_N - S_N f \in P_N$, by the Pythagorean theorem,

$$\|p_N - f\|_{L^2_{\text{per}}} = \|S_N f - f\|_{L^2_{\text{per}}} + \|p_N - S_N\|_{L^2_{\text{per}}} \geq \|S_N f - f\|_{L^2_{\text{per}}}.$$

$\square$

Corollary 29.8. If there exists $p_N \in P_N$ for all $N \in \mathbb{N}$ such that $p_N \to f$ in L^2_{per}, then $S_N f \to f$ in L^2_{per}.

We can attempt to explicitly compte $S_N f$:

$$S_N f(x) = \sum_{n=-N}^{N} \mathcal{F} f(n) e^{2\pi i n x} = \int_0^1 \left(f(y) \sum_{n=-N}^{N} e^{-2\pi i n (x-y)} \right) dy.$$

Definition 29.9 (Dirichlet Kernel)**.** Define $D_N(x) := \sum_{n=-N}^{N} e^{2\pi i n x}$.

Remark 29.10. We can explicitly write out D_N as:

$$D_N(x) = \frac{\sin(\pi(2N+1)x)}{\sin(\pi x)}.$$

Proof. This was done in Homework 11. $\square$

So we can write $S_N f = f * D_N$. Unfortunately, D_N is note an approximate identity. (In fact, $\int_0^1 |D_N| x \approx \ln N$). So we must try a different approach.

Definition 29.11 (Cesàro Sum)**.** Define $\sigma_N f = \frac{1}{N} \sum_{n=0}^{N-1} S_N f$. Certainly, $\sigma_N f = \left(\frac{1}{N} \sum_{n=0}^{N-1} D_N \right) * f$, by linearity.

Definition 29.12 (Fejér Kernel)**.** Define $F_N = \frac{1}{N} \sum_{n=0}^{N-1} D_N$.

Remark 29.13. We can explicitly write out F_N as:

$$F_N = \frac{1}{N} \left(\frac{\sin(N\pi x)}{\sin(\pi x)} \right)^2.$$

Proof. This was done in Homework 11. $\square$

Claim 29.14. F_N is an approximate identity.

Proof. This was done in Homework 11. $\square$

Corollary 29.15. If $f \in L^p_{\text{per}}$, then $\sigma_N f \to f$ in L^p_{per}. If f is continuous, then $\sigma_N f \to f$ uniformly.

Proof. The first statement follows from Proposition 28.8. $\square$

Corollary 29.16. If $f \in L^2$, then $S_N f \to f$ in L^2.

Proof. Since $\sigma_N f \in P_N$ for all N, the result follows by Corollary 29.8 and Corollarly 29.15. $\square$

Corollary 29.17 (Riemann-Lebesgue)**.** If $f \in L^1_{\text{per}}$ then $\mathcal{F}f(n) \to 0$ as $|n| \to \infty$.

Proof. Let $\epsilon > 0$. Choose large N such that $\|\sigma_N f - f\| < \epsilon$. Let $g = f - \sigma_N f$. Then $f = g + \sigma_N f$. Then $\mathcal{F}(n) = \mathcal{F}g(n) + \mathcal{F}(\sigma_N f)(n)$. For $|n| > N$, $\mathcal{F}(\sigma_N f)(n) = 0$. Also $|\mathcal{F}g(n)| = \left|\int_0^1 ge_n, dx\right| \leq \int_0^1 g\, dx = \|g\|_1 < \epsilon$. Hence $|\mathcal{F}f(n)| < \epsilon$. Taking $\epsilon \to 0$ gives the result. $\square$

30 November 11, 2013

Corollary 30.1 (Parseval's Identity)**.** If $f \in L^2_{\text{per}}$, then $\|f\|_{L^2_{\text{per}}} = \|\mathcal{F}f\|_{\ell^2(\mathbb{Z})}$. Further, the map $\phi : f \mapsto \mathcal{F}$ is a bijective linear isometry from L^2_{per} to $\ell^2(\mathbb{Z})$.

Proof. Since the e_n's are orthogonal to each other, by Pythagorean's Theorem,

$$\|S_N f\|^2_{L^2_{\text{per}}} = \sum_{n=-N}^{N} |\mathcal{F}f(n)|^2 \|e_n\|^2_{L^2_{\text{per}}} = \sum_{n=-N}^{N} |\mathcal{F}f(n)|^2.$$

Taking $N \to \infty$, since $S_N f \to f$ in L^2, we have $\|f\|^2_{L^2_{\text{per}}} = \|\mathcal{F}f\|^2_{\ell^2(\mathbb{Z})}$.

This also proves that ϕ is injective. Clearly ϕ is linear. It remains to show that ϕ is surjective. Take $\{a_n\} \in \ell^2(\mathbb{Z})$. Define $f = \sum_{n\in\mathbb{N}} a_n e_n$. For $0 \le M \le N$, by orthogonality, we have

$$\left\| \sum_{M \le |n| \le N} a_n e_n \right\|^2_{L^2_{\text{per}}} = \sum_{M \le |n| \le N} \|a_n e_n\|_{L^2_{\text{per}}} = \sum_{M \le |n| \le N} |a_n|^2.$$

We know that $\sum_{M\le|n|\le N} |a_n|^2$ is a convergent Cauchy sequence, so that the sum $\sum_{M\le|n|\le N} \|a_n e_n\|_{L^2_{\text{per}}}$ is Cauchy and convergent. So, $\sum_{n\in\mathbb{Z}} \|a_n e_n\|_{L^2_{\text{per}}} < \infty$, which implies that $\sum_{n\in\mathbb{Z}} a_n e_n$ converges in L^2_{per}. Hence, $f \in L^2_{\text{per}}$. □

Definition 30.2 (Fourier Series of Measure)**.** Say μ is a finite measure on $[0, 1]$. Define

$$\mathcal{F}\mu(n) = \int_0^1 \overline{e_n}, d\mu.$$

Note 30.3. If $d\mu = f d\lambda$ and $\tau_1 f = f$, then $\mathcal{F}\mu(n) = \int_0^1 \overline{e_n} f \, d\lambda = \mathcal{F}f(n)$.

Lemma 30.4. If $f \in L^1_{\text{per}}$ and $\xi \in \mathbb{R}$, then $\mathcal{F}(\tau_\xi f)(n) = \overline{e_n(\xi)}\mathcal{F}f(n)$.

Proof. By the periodicity of f,

$$\mathcal{F}(\tau_\xi f)(n) = \int_0^1 f(x-\xi) e^{-2\pi i n x} \, dx = \int_0^1 f(x) e^{-2\pi i n (x+\xi)} = \overline{e_n(\xi)}\mathcal{F}f(n).$$

□

Lemma 30.5 (Riemann Lebesgue)**.** If $f \in L^1_{\text{per}}$, then $\mathcal{F}f(n) \to 0$ as $|n| \to \infty$.

Proof. Let $\xi = \frac{1}{2n}$. Then $e_n(\xi) = -1$. Hence, by Lemma 30.4, $\mathcal{F}(\tau_\xi f)(n) = -\mathcal{F}f(n)$, so that

$$2\mathcal{F}f(n) = \mathcal{F}f(n) - \mathcal{F}(\tau_\xi f)(n) = \mathcal{F}\left(f - \tau_{\frac{1}{2n}} f\right)(n).$$

Hence, by Hölder's inequality, $2|\mathcal{F}f(n)| \le \left\| f - \tau_{\frac{1}{2n}} f \right\|_{L^1_{\text{per}}}$. Taking $n \to \infty$ gives the result. □

31 November 13, 2013

Definition 31.1 (Weak Derivative)**.** We say $f \in L^p_{\text{per}}$ has a weak derivative Df, which is periodic with period 1, if for every $\phi \in C^\infty_{\text{per}}$,

$$\int_0^1 f\phi'\,dx = -\int_0^1 Df \cdot \phi\,dx.$$

Remark 31.2. If $f \in C^1$, then Df exists and $Df = f'$. The converse if false, however. Consider any periodic function that looks like the absolute-value function.

Lemma 31.3. If $f \in L^2_{\text{per}}$ and $Df \in L^1_{\text{per}}$, then $\mathcal{F}(Df)(n) = 2\pi i n \mathcal{F}f(n)$.

Proof. Since $e^{-2\pi i n x} \in C^\infty_{\text{per}}$, so that

$$\begin{aligned}
\mathcal{F}(Df)(n) &= \int_0^1 Df(x)\overline{e_n}(x)\,dx = \int_0^1 Df(x)e^{-2\pi i n x}\,dx \\
&= -\int_0^1 f(x)\left(-2\pi i n e^{-2\pi i n x}\right)\,dx = 2\pi i n \mathcal{F}f(n).
\end{aligned}$$

□

Proposition 31.4. If $f, Df \in L^2_{\text{per}}$, then

$$\sum_{n\in\mathbb{Z}} (1+n^2)|\mathcal{F}f(n)|^2 < \infty.$$

In other words, $\left\{\mathcal{F}f(n)(1+n^2)^{\frac{1}{2}}\right\}_n \in \ell^2(\mathbb{Z})$.

Proof. By Parseval's identity, $\sum_{n\in\mathbb{Z}} |\mathcal{F}f(n)|^2 = \|f\|^2_{L^2_{\text{per}}} < \infty$. By Lemma 31.3, and applying Parseval's identity again,

$$\sum_{n\in\mathbb{Z}} |n\mathcal{F}f(n)|^2 = \sum_{n\in\mathbb{Z}} \left|\frac{1}{2\pi i}\mathcal{F}(Df)(n)\right|^2 = \frac{1}{2\pi^2}\|Df\|_{L^2_{\text{per}}} < \infty.$$

And the result follows. □

Corollary 31.5. If $f \in L^2_{\text{per}}$, $D^i f$ exists for $i = 1, \ldots, k$ for some $k \in \mathbb{N}$, and $D^i f \in L^2_{\text{per}}$ for each i, then

$$\sum_{n\in\mathbb{Z}} \left(1+n^2\right)^k |\mathcal{F}f(n)|^2 < \infty.$$

In other words, $\left\{\mathcal{F}f(n)(1+n^2)^{\frac{k}{2}}\right\}_n \in \ell^2(\mathbb{Z})$.

Definition 31.6 (Sobolev Space)**.** Let $s \geq 0$. Define

$$H^s_{\text{per}} = \left\{ f \in L^2_{\text{per}} \,\middle|\, \sum_{n \in \mathbb{Z}} \left(1+n^2\right)^s |\mathcal{F}f(n)|^2 < \infty \right\}$$

and define the norm of $f \in H^s_{\text{per}}$ as

$$\|f\|_{H^s_{\text{per}}} = \sum_{n \in \mathbb{Z}} \left(1+n^2\right)^s |\mathcal{F}f(n)|^2.$$

Remark 31.7. If $s \in \mathbb{N}$, then $H^s_{\text{per}} = \left\{ f \in L^2_{\text{per}} \,\middle|\, D^s f \text{ exists and } D^s f \in L^2 \right\}$.

Theorem 31.8 (Sobolev Embedding)**.** If $f \in H^s_{\text{per}}$ and $s > \frac{1}{2}$, then f is continuous and $\|f\|_\infty \leq C \|f\|_{H^s_{\text{per}}}$ for some constant C independent of f.

Proof. We have

$$\sum_{n \in \mathbb{Z}} |\mathcal{F}f(n)| = \sum_{n \in \mathbb{Z}} \left(1+n^2\right)^{\frac{s}{2}} |\mathcal{F}f(n)| \cdot \frac{1}{(1+n^2)^{\frac{s}{2}}}.$$

By Cauchy-Schwartz, this implies

$$\sum_{n \in \mathbb{Z}} |\mathcal{F}f(n)| \leq \left(\sum_{n \in \mathbb{Z}} \left(1+n^2\right)^s |\mathcal{F}f(n)|^2 \right)^2 \left(\sum_{n \in \mathbb{Z}} \frac{1}{(1+n^2)^s} \right)^{\frac{1}{2}}.$$

The first sum on the right is equal to $\|f\|_{H^s_{\text{per}}} < \infty$, and the second sum is equal to some finite C since $s > \frac{1}{2}$. So, $\sum_{n \in \mathbb{Z}} |\mathcal{F}f(n)| \leq C \|f\|_{H^s_{\text{per}}} < \infty$. So, by Lemma 18.2, it follows that $\sum_{n=-N}^{N} \mathcal{F}f(n) e_n(x) \to f$ uniformly. Since each $\mathcal{F}f(n)e_n(x)$ is continuous, f is continuous as well. Finally, by the countable triangle inequality, $|f| \leq \sum_{n \in \mathbb{Z}} |\mathcal{F}f(n)| \leq C \|f\|_{H^s_{\text{per}}}$. □

Proposition 31.9 (Relich Lemma)**.** If $f_n \in H^t_{\text{per}}$ for each $n \in \mathbb{N}$ such that $\sup_{n \in \mathbb{N}} \|f_n\|_{H^t_{\text{per}}} < \infty$, then for every $s < t$, there exists a subsequence $\{f_{n_k}\}_{k \in \mathbb{N}}$ that is convergence in H^s_{per}.

Proof. This was done in Homework 12. □

32 November 16, 2013

Definition 32.1 (Maximal Function)**.** Let μ be a finite signed Borel (and thus regular) measure on $\mathbb{R}^d$. Define

$$M\mu(x) = \sup_{r>0} \frac{|\mu|(B(x,r))}{\lambda(B(x,r))}.$$

If $d\mu = f d\lambda$, then we also write $Mf(x) = M\mu(x)$.

We would like an estimate of the form $\|Mf\|_1 \leq C\|f\|_1$. However, this is false. Take $f = \chi_{[-1,1]}$. Then

$$Mf = \begin{cases} 1 & x \in |x| < 1 \\ \frac{1}{|x|+1} & |x| \geq 1 \end{cases}.$$

Lemma 32.2 (Vitali Coverling Lemma)**.** Let $A \subseteq \mathbb{R}^d$, $\{B_1, \dots, B_N\}$ be a collection of balls that covers A. Then there exists a disjoint subcollection $\{B_{n_1}, \dots, B_{n_k}\}$ such that $A \subseteq \bigcup_{i=1}^k 3B_{n_i}$.

Proof. Choose B_{n_1} to be the ball of the largest radius. Among all balls that don't intersect with B_{n_1}, let B_{n_2} be the ball the the largest radius. Recursively repeat this procedure. This procedure terminates since there are only a finite number of balls to choose from. Let the balls chosen be $B_{n_1}, \dots, B_{n_k}$.

Clearly $B_{n_1}, \dots, B_{n_k}$ are disjoint. If B_j is not B_{n_i} for any i, then there exists an i_0 such that $B_j \cap B_{n_{i_0}} \neq \emptyset$. By choice of the B_{n_i}'s, it follows that $r_{n_{i_0}} > r_j$, where $r_{n_{i_0}}$ is the radius of $B_{n_{i_0}}$ and r_j is the radius of B_j. Hence, by the triangle inequality, $3B_{n_{i_0}} \supseteq B_j$. $\square$

Proposition 32.3. Let μ be a finite signed Borel measure. Then for all $\alpha > 0$,

$$\lambda\left(\{x \in X \mid M\mu(x) > \alpha\}\right) \leq \frac{3^d}{\alpha}\|\mu\|.$$

Proof. Without loss of generality, we may assume μ is a positive measure. Fix $\alpha > 0$. Let $S = \{x \in X \mid M\mu(x) > \alpha\}$. Since μ is finite and Borel, μ is regular. Hence, it suffices to show that $\lambda(K) \leq \frac{3^d}{\alpha}\|\mu\|$ for all compact $K \subseteq S$.

Choose any compact $K \subseteq S$. For each $x \in K$, $M\mu(x) > \alpha$; so we can find $r(x) > 0$ such that $\mu(B(x, r(x))) > \alpha\lambda(B(x, r(x)))$. Now, $\{B(x, r(x))\}_{x \in K}$ is a covering of K by balls. By compactness, we can find a finite subcover $\{B(x_n, r(x_n))\}_{1 \leq n \leq N}$ for some $N \in \mathbb{N}$. By the Vitali covering lemma, we can find a smaller subset $\{B(x_{n_m}, r(x_{n_m}))\}_{1 \leq m \leq M}$ of disjoint balls, where $M \in \mathbb{N}$, such that $K \subseteq \bigcup_{m=1}^M B(x_{n_m}, 3r(x_{n_m}))$. For brevity, let $B_m = B(x_{n_m}, r(x_{n_m}))$.

Hence,

$$\begin{aligned}\lambda(K) \le \sum_{m=1}^{M} \lambda\left(3B_m\right) = 3^d \sum_{m=1}^{M} \lambda(B_m) \\ < \frac{3^d}{\alpha} \sum_{m=1}^{M} \mu(B_m) = \frac{3^d}{\alpha}\mu\left(\bigcup_{m=1}^{M} B_m\right) \le \frac{3^d}{\alpha}\left\|\mu\right\|.\end{aligned}$$

□

Corollary 32.4. If $f \in L^1$, then for all $\alpha > 0$,

$$\lambda\left(\{x \in X \mid Mf(x) > \alpha\}\right) \le \frac{3^d}{\alpha}\left\|f\right\|_1.$$

33 November 18, 2013

Proposition 33.1 (Lebesgue Differentiation)**.** If $f \in L^1$, then for a.e. $x \in \mathbb{R}^d$,

$$\lim_{r\to\infty} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f \, d\lambda = f(x).$$

Proof. It is enough to show that for a.e. x,

$$\Omega f(x) := \limsup_{r\to\infty} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dy = 0.$$

Observe that if g is continuous, then $\Omega g = 0$. Fix $\epsilon > 0$. Let $\alpha > 0$. Consider $S := \left\{x \in \mathbb{R}^d \,\middle|\, \Omega f > \alpha\right\}$. By density of continuous functions in L^1, we can find a continuous g such that $\|f - g\|_1 < \epsilon$. Let $h = f - g$. Then

$$\Omega f(x) = \Omega(g + h)(x) \le \Omega g(x) + \Omega h(x) = \Omega h(x).$$

Hence

$$\begin{aligned}
\lambda(S) \le \lambda(\{\Omega h > \alpha\}) &\le \lambda(\{Mh + |h| > \alpha\}) \\
&\le \lambda\left(\left\{Mh > \frac{\alpha}{2}\right\}\right) + \lambda\left(\left\{|h| > \frac{\alpha}{2}\right\}\right) \\
&\le \frac{2 \cdot 3^d}{\alpha} \|h\|_1 + \frac{2}{\alpha} \|h\|_1 \qquad (4) \\
&< \frac{2 \cdot 3^d + 2}{\alpha} \epsilon,
\end{aligned}$$

where (4) follows from Corollary 32.4 and Chebyshev's inequality. Hence, taking $\epsilon \to 0$, it follows that $\lambda(S) = 0$. Taking $\alpha \to 0$, we have $\lambda(\{\Omega f > 0\}) = 0$. Hence, $\Omega f = 0$ a.e., and the result follows. ☐

Definition 33.2 (Derivative of Measure)**.** Let μ be a positive σ-finite Borel measure or a signed finite Borel measure on $\mathbb{R}^d$. For simplicity, we assume that μ is positive, finite and Borel. Define

$$D\mu(x) = \lim_{r\to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))}.$$

Note 33.3. If $\mu \ll \lambda$, by Radon-Nikodým, there exists an $f \in L^1(\mathbb{R}^d, \lambda)$ such that $d\mu = f d\lambda$. So

$$D\mu(x) = \lim_{r\to 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f \, d\lambda = f(x)$$

for a.e. $x \in \mathbb{R}^d$ by Lebesgue differentiation. So $D\mu = \frac{d\mu}{d\lambda}$ for a.e. $x \in \mathbb{R}^d$.

Example 33.4. What happens when $\mu \perp \lambda$? For an example, consider $\mu = \delta_0$, the delta measure with mass of 1 concentrated at 0. Then

$$D\delta_0(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}.$$

Proposition 33.5. If μ is finite and Borel, then

1. if $\mu \ll \lambda$, then $D\mu = \frac{d\mu}{d\lambda}$ λ-a.e.;
2. if $\mu \perp \lambda$, then $D\mu = 0$ λ-a.e. and $D|\mu| = \infty$ μ-a.e.

Proof. Suppose $\mu \ll \lambda$. By Note 33.3, we already have $D\mu = \frac{d\mu}{d\lambda}$ λ-a.e.

Suppose $\mu \perp \lambda$. Then we can find $N \in \mathcal{B}\left(\mathbb{R}^d\right)$ such that $\lambda(N) = 0$ and $\mu(N^c) = 0$. Fix $\epsilon > 0$. Since μ is finite and Borel, $|\mu|$ is regular. So we can find compact $K \subseteq N$ such that $|\mu|(N - K) < \epsilon$.

Let $\sigma(A) = \mu(A \cap K)$ and $\nu(A) = \mu(A \cap K^c)$. Observe that $\|\nu\| < \epsilon$. For every $x \notin K$, we can find an small enough $r > 0$ such that $B(x, r) \subseteq K^c$, so that $\sigma(B(x, r)) = 0$. $D\sigma(x) = 0$ on K^c. Since $\lambda(K) = 0$, it follows $D\sigma(x) = 0$ λ-a.e. Let

$$\overline{D}\mu(x) := \limsup_{r \to 0} \frac{\mu(B(x, r))}{\lambda(B(x, r))}.$$

Then $\overline{D}\mu \leq \overline{D}\sigma + \overline{D}\nu = \overline{D}\nu$ λ-a.e., so that

$$\lambda(\{\overline{D}\mu > \alpha\}) \leq \lambda(\{\overline{D}\nu > \alpha\}) \leq \lambda(\{M\nu > \alpha\}) \leq \frac{3^d}{\alpha} \|\nu\| < \frac{3^d}{\alpha}\epsilon.$$

Taking $\epsilon \to 0$, it follows that $D\mu = 0$ λ-a.e.

The proof that $D|\mu| = \infty$ μ-a.e. was done in Homework 13. $\square$

34 November 20, 2013

Corollary 34.1. Let μ be a finite Borel measure. By Lebesgue decomposition, we can write $\mu = \mu_{\text{ac}} + \mu_{\text{s}}$ where $\mu_{\text{ac}} \ll \lambda$ and $\mu_{\text{s}} \perp \lambda$. Then $D\mu$ exists λ-a.e. and $D\mu = \frac{d\mu}{d\lambda}$ λ-a.e.

Corollary 34.2. Let $A \in \mathcal{L}(\mathbb{R}^d)$ be given. Then for λ-a.e. x,

$$\lim_{r \to 0} \frac{\lambda(A \cap B(x,r))}{\lambda(B(x,r))} = \chi_A(x).$$

Remark 34.3. Recall H_α is the Hausdorff measure of dimension α. Define $C(\alpha) = \frac{\pi^\alpha/2}{\Gamma\left(1+\frac{\alpha}{2}\right)}$. Let $A \subseteq \mathbb{R}^d$ be given with $H_\alpha(A) \in (0, \infty)$. Then

1. $$\lim_{r \to 0} \frac{H_\alpha(A \cap B(x,r))}{C(\alpha) r^\alpha} = 0 \text{ for } H_\alpha\text{-a.e. } x \notin A.$$

2. For a.e. $x \in A$, we have

$$\limsup_{r \to \infty} \frac{H_\alpha(A \cap B(x,r))}{C(\alpha) r^\alpha} \in \left[\frac{1}{2^a}, 1\right]$$

3. There exists A with $H_\alpha < \infty$ and $0 < \alpha < d$ such that

$$\limsup_{r \to \infty} \frac{H_\alpha(A \cap B(x,r))}{C(\alpha) r^\alpha} < 1 \ H_\alpha \text{ a.e..}$$

and

$$\liminf_{r \to \infty} \frac{H_\alpha(A \cap B(x,r))}{C(\alpha) r^\alpha} = 0$$

Lemma 34.4 (Differentiation of Functions)**.** We restrict our attention to $\mathbb{R}$. Pick $f \in L^1$. Define $F(x) = \int_0^x f \, dx$. Then $F(x)$ is differentiable almost everywhere and $F' = f$ almost everywhere.

Proof. The following is a "proof." The reader should attempt to fix it.

$$F' = \lim_{h \to 0} \frac{F(x+h) - F(x-h)}{2h} = \frac{1}{\lambda(B(x,h))} \int_{B(x,h)} f \, d\lambda = f(x) \text{ a.e.}$$

$\square$

The goal here eventually is to prove the Fundamental Theorem of Calculus; we would like a result of the form

$$\int_a^b f' \, d\lambda = f(b) - f(a).$$

Note 34.5. f being differentiable almost everywhere does not imply that $f' \in L^1[a,b]$. For an example, take $f(x) = \ln x$. We have $f' = \frac{1}{\lambda} \notin L1[0,1]$.

Note 34.6. f being differentiable almost everywhere and $f' \in L^1$ does not imply $f(b) - f(a) = \int_a^b f'\,dx$. For an example, let f be the Devil's Staircase.

Definition 34.7 (Absolute Continuity)**.** We say a function $f : [a, b] \to \mathbb{R}$ is *absolutely continuous* if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $\{(x_i, y_i)\}_{i\in\mathbb{N}}$ is a collection of disjoint intervals such that $\sum_{i\in\mathbb{N}} |x_i - y_i| < \delta$, then $\sum_{i\in\mathbb{N}} |f(x_i) - f(y_i)| < \epsilon$.

Proposition 34.8. If f is absolutely continuous, then f is continuous.

Note 34.9. Continuity does not imply absolute continuity. For an example, consider the Cantor function.

Claim 34.10. Let $f \in L^1[a, b]$. Define $F(x) = \int_a^x f\,dx$. Then F is absolutely continuous.

Proof. Fix $\epsilon > 0$. f is uniformly integrable, so there exists a $\delta > 0$ such that for all $A \in \Sigma$ such that $\mu(A) < \delta$, we have $\int_A |f|\,d\lambda < \epsilon$. Consider any collection of disjoint intervals $\{(x_i, y_i)\}_{i\in\mathbb{N}}$ such that $\sum_{i\in\mathbb{N}} |x_i - y_i| < \delta$. Let $A = \bigcup_{i\in\mathbb{N}}(x_i, y_i)$, so that $\mu(A) < \delta$. Then

$$\sum_{i\in\mathbb{N}} |F(x_i) - F(y_i)| = \sum_{i\in\mathbb{N}} \left| \int_{x_i}^{y_i} f\,dx \right| \le \sum_{i\in\mathbb{N}} \int_{(x_i,y_i)} |f|\,d\lambda = \int_A |f|\,d\lambda < \epsilon.$$

□

35 November 22, 2013

Today's goal is to eventually prove the Fundamental Theorem of Calculus.

Lemma 35.1. Assume $f : [a,b] \to \mathbb{R}$ is absolutely continuous. Suppose f is strictly increasing. Then f is differentiable almost everywhere, $f \in L^1$, and for every $x, y \in [a,b]$ with $x < y$, we have

$$\int_x^y f' \, d\lambda = f(y) - f(x).$$

Proof. Define $\mu(A) = \lambda(f(A))$. Since f injective, it is easy to check that μ is a measure. The reader should check that μ is a regular measure.

We claim that $\mu \ll \lambda$. Let $A \subseteq [a,b]$ such that $\lambda(A) = 0$. Pick $K \subseteq A$ compact. We need to show that $\mu(A) = 0$. By regularity of μ, it suffices to show that $\mu(K) = 0$ for all $K \subseteq A$. Pick $\epsilon > 0$. Pick δ in the definition of absolute continuity of f. Then there exist a finite collection of disjoint intervals $\{(x_i, y_i)\}$ such that $K \subseteq \bigcup_i (x_i, y_i)$ and $\sum_i |x_i - y_i| < \delta$. Thus $\sum_i |f(x_i) - f(y_i)| < \epsilon$ and $\mu\left(\bigcup_i (x_i, y_i)\right) < \epsilon$. So $\mu(K) < \epsilon$. Taking $\epsilon \to 0$, we have $\mu(K) = 0$. Thus $\mu \ll \lambda$.

By Radon-Nikodým, there exists $g \in L^1[a,b]$ such that $d\mu = g \, d\lambda$. Then for all $x, y \in [a,b]$ with $x < y$, $\mu((x,y)) = f(y) - f(x)$. Thus, $\int_x^y g \, d\lambda = f(y) - f(x)$.

Since $f(x) = f(a) + \int_a^x g \, d\lambda$, Lemma 34.4 implies that f is differentiable almost everywhere and $f' = g$. □

Lemma 35.2. Assume $f : [a,b] \to \mathbb{R}$ is absolutely continuous. Suppose f is increasing. Then f is differentiable almost everywhere, $f \in L^1$, and for every $x, y \in [a,b]$ with $x < y$, we have

$$\int_x^y f' \, d\lambda = f(y) - f(x).$$

Proof. Let $g(x) = f(x) + x$. Then g is strictly increasing and absolutely continuous. Lemma 35.1 implies that g is differentiable almost everywhere and $g' \in L^1[a,b]$ and $g(y) - g(x) = \int_x^y g' \, d\lambda$. Then $g(x) = g(a) + \int_a^x g' \, d\lambda$ implies $f(x) + x = f(a) + a + \int_a^x g' \, d\lambda$. Thus f is differentiable almost everywhere and $f' = g' - 1$. □

Before we can finally prove the more general form of the Fundamental Theorem of Calculus, we prove a claim.

Claim 35.3. Assume $f : [a,b] \to \mathbb{R}$ is absolutely continuous. There exist functions g and h such that $f = g - h$ where g and h are absolutely continuous and increasing

Proof. Let $F(x)$ be the variation of f on $[a,x]$; that is,

$$F(x) = \sup \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| \text{ where } a = x_0 < x_1 < \cdots < x_N = x.$$

We claim F is also finite and absolutely continuous.

Pick $\epsilon = 1$. Then we can find $\delta > 0$ such that if $\{(x_i, y_i)\}$ is a collection of disjoint intervals such that $\sum_i |x_i' - y_i'| < \delta$, then $\sum_i |f(x_i') - f(y_i')| < \epsilon = 1$. Any partition $a = x_0 < x_1 < \ldots < x_N = x$ of $[a, x]$ with mesh size less than δ has $\sum_{i=1}^N |f(x_i) - f(x_{i-1})| < \frac{x-a}{\delta} \cdot 1 < \infty$. Note that

$$F(y) - F(x) = \sup \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| \text{ where } x = x_0 < x_1 < \cdots < x_N = y.$$

Set $g = F + \frac{1}{2}f$ and $h = F - \frac{1}{2}f$. The reader should check that g and h are increasing and absolutely continuous. $\square$

Theorem 35.4 (Fundamental Theorem of Calculus)**.** Assume $f : [a, b] \to \mathbb{R}$ is absolutely continuous. Then f is differentiable almost everywhere, $f \in L^1$, and for every $x, y \in [a, b]$ with $x < y$, we have

$$\int_x^y f' \, d\lambda = f(y) - f(x).$$

Proof. The function f is absolutely continuous. We need to show that $f' \in L^1[a, b]$ and $f(x) = f(a) + \int_a^x f' \, d\lambda$. By Claim 35.3, we can find absolutely continuous and increasing functions g and h such that $f = g - h$. By Lemma 35.2 and linearity, the result follows. $\square$

Remark 35.5. f has bounded variation if and only if $f = g - h$ where g and h are increasing.

Remark 35.6. If f absolutely continuous, then f has bounded variation.

36 November 25, 2013

Today's goal is to study change of variables.

Theorem 36.1 (Change of Variables)**.** Let $U, V \subseteq \mathbb{R}^d$ be open. Let $\phi : U \to V$ be C^1 and bijective. Let $f \in L^1(V)$. Then

$$\int_V f\, d\lambda = \int_U f \circ \phi |\det \nabla\phi|\, d\lambda,$$

where $\nabla\phi$ is the Jacobian of ϕ, which is equal to $(\partial_j \phi_i)_{ij}$.

Proof. Define $\mu(A) = \lambda(\phi(A))$. Since ϕ bijective, μ defines a measure on V. From the questions on pull-back measures from the Homework, we know that

$$\int_U f \circ \phi\, d\mu = \int_V f\, d\lambda,$$

To prove the theorem, only need to show that $d\mu = |\det \nabla\phi|\, d\lambda$.

Lemma 36.2. $\mu \ll \lambda$

Lemma 36.3. $d\mu/d\lambda = |\det \nabla\phi|$.

Note that these two lemmas together imply the Theorem.

Proof of Lemma 36.3. Because $\mu \ll \lambda$,

$$\frac{d\mu}{d\lambda} = \lim_{r\to 0} \frac{\mu(B(x,r))}{\lambda(B(x,v))} = \lim_{r \to 0} \frac{\lambda(\phi(B(x,r)))}{\lambda(B(x,r))}$$

We need to show that the limit on the right-hand side is equal to $|\det \nabla\phi|$.

Step 1: Without loss of generality, assume $x = 0$ and $\phi(x) = 0$. Suppose $\nabla\phi(0) = I$. Then $\lim_{r\to 0} \frac{\lambda(\phi(B(x,r)))}{\lambda(B(x,r))} = 1$.

Proof. Let $\epsilon > 0$. Then there exists r_0 such that $|x| < r_0 \implies \|\nabla\phi(x) - I\| < \epsilon$, where for $T \in \mathbb{R}^{d\times d}$ we define $\|T\| = \sup \frac{|Tx|}{\|x\|}$.

Let $g(x) = \phi(x) - x$. Note that $|g_j(x)| \le \epsilon|x|$, because

$$g_j(x) = g_j(x) - g_j(0) =_{MVT} (\nabla g)_j(y) \cdot x \le \|\nabla g\||x| \le \epsilon|x| \text{ when } |x| < r_0$$

So $\phi(x) \in B(0, (1_\epsilon)|x|) - B(0, (1-\epsilon)|x|)$.

Thus, for $r < r_0$,

$$(1-\epsilon)^d \lambda(B(0,r)) \le \lambda(\phi(B(0,r))) \le (1_\epsilon)^d \lambda(B(0,r))$$

where the middle expression follows from the bijectivity of ϕ.

Taking $\epsilon \to 0$, we have proven Step 1. $\square$

Step 2: Say $\nabla\phi(0) = T$ and T is invertible. Then

$$\lim_{r\to 0} \frac{\lambda(\phi(B(0,r)))}{\lambda(B(0,r))} = |\det T|.$$

Proof. Set $\psi = T^{-1} \circ \phi$. Then $\nabla\psi(0) = I$. Step 1 implies that

$$\lim_{r\to 0} \frac{\lambda(\psi(B(0,r)))}{\lambda(B(0,r))} = 1.$$

Thus,

$$\lim_{r\to 0} \frac{\lambda(T^{-1}(\phi(B(0,r))))}{\lambda(B(0,r))} = |\det T^{-1}| \lim_{r\to 0} \frac{\lambda(\phi(B(0,r)))}{\lambda(B(0,r))}.$$

□

Step 3: What if $\nabla\phi(0)$ is not invertible? Then $|\det \nabla\phi(0)| = 0$, so we need to show that $\lim_{r\to 0} \frac{\lambda(\phi(B(0,r)))}{\lambda(B(0,r))} = 0$. Let $T = \nabla\phi(0)$. We know $\det T = 0$ and $T(\mathbb{R}^d)$ is a subset of a $(d-1)$-dimensional subspace. You check the rest. The idea is that $\phi(x)$ lies within ϵ of that subspace. □

Proof of Lemma 36.2. **Step 1**: μ is regular (You check: $U = \cup_{n=1}^{\infty} K_n$, K_n compact)

Step 2: Pick $A \subseteq U$ and $\lambda(A) = 0$. We need to show that $\mu(A) = 0$.

By Step 1, it suffices to show that $\mu(K) = 0$ for all $K \subseteq A$ compact. (Note: As you will see by the end of the proof, we do not need that μ is regular. We can use the infinite version of the Vitali covering lemma.)

Pick $\epsilon > 0$. We know that there exists $U \supseteq K$ with $\lambda(U) < \epsilon$. Since K is compact, there exists $c < \infty$ such that $\sup_{x\in K} |Vert\nabla\phi(x)\| < c$. For all X in K, there exists $r(x)$ such that $B(x, r(x)) \subseteq U$ and

$$\sup_{x\in B(x,r(x))} \|\nabla\phi(x)\| < 2c.$$

Compactness lets us pass to a finite subcover. Then by Vitali, there exist $x_1, \ldots, x_n \in K$ such that $\{B(x_i, r(x_i))\}_{1\le i\le N}$ disjoint and $J \subseteq \cup_{i=1}^{N} B(x_i, r(x_i))$.

You check: $\mu(B(x_i, r(x_i))) \le 10c\lambda(B(x_i, r(x_i)))$ implies that

$$\mu(K) \le \mu(\cup B(x_i, 3r_i)) \le 3^d 10c\lambda(U) \le 3^d 10c\epsilon.$$

□

The two lemmas together imply the theorem. □

37 December 2, 2013

Fourier Transform

Definition 37.1. Let $f \in L^1(\mathbb{R}^d)$ be given. Define the Fourier transform $\hat{f}$ by

$$\hat{f}(\zeta) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i <x,\zeta>}$$