720 Measure Theory: Final.

Thu, Dec 17^{th}

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 3 hours. The exam has a total of 7 questions and 70 points.*
- *You may use without proof any result that has been proved in class or on the homework, unless you are explicitly instructed otherwise. You must, however, **CLEARLY** state the result you are using.*
- *The questions are roughly in order of difficulty; but depending on your intuition you might find some questions easier than others.*

Unless otherwise stated, we always assume the underlying measure space is (X, Σ, μ) and μ is a positive measure. The Lebesgue measure on $\mathbb{R}^d$ will be denoted by λ.

1. *(10 points)* For any $C \subseteq [0,1]$ closed, and any $\varepsilon > 0$ do there exist finitely many closed intervals $I_1, \dots, I_N$ such that $\cup_1^N I_i \subseteq C$ and $\lambda(C - \cup_i I_i) < \varepsilon$)? Prove, or provide a counter example.
2. *(10 points)* Let μ, ν be two finite positive measures on X. Suppose $\nu \ll \mu$, and let $f = \frac{d\nu}{d\mu}$ be the Radon Nikodym derivative of ν with respect to μ. Find a necessary and sufficient condition on f that guarantees $\mu \ll \nu$. Prove it.
3. *(10 points)* Fix $p \in [1, \infty]$. Suppose there exists $q \in [1, \infty]$ and $c \in (0, \infty)$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)$ we have $\|\hat{f}\|_q \leqslant c\|f\|_p$. Find q in terms of p.
4. *(10 points)* Let $f \in L^1(\mathbb{R}^d)$. Compute $\lim_{|y| \to 0} \int_{\mathbb{R}^d} |f(x-y) - f(x)| \, d\lambda(x)$. Prove your answer. [This is a special case of a question on your homework. Please provide a self contained proof here, and don't simply say "done on homework". You may, however, use without proof other standard results in class / homework that *do not* rely on this problem.]
5. *(10 points)* We proved the following version of Vitali's theorem in class: If (1) $f_n, f \in L^1(\mu)$, (2) $(f_n) \to f$ in measure, (3) $\{f_n\}$ is uniformly integrable, and (4) $\forall \varepsilon > 0$, $\exists E \in \Sigma$ such that $\mu(E) < \infty$ and $\int_{E^c} |f_n| \, d\mu < \varepsilon$, then $(f_n) \to f$ in $L^1(\mu)$. Does the theorem still hold if we drop the assumption $f \in L^1$? (We of course retain the assumption $f_n \in L^1(\mu)$, and the other three assumptions.) Prove it, or find a counter example.
6. In this question we assume all functions are periodic with period 1 (i.e. $\tau_1 f = f$). Recall

$$H^1_{\text{per}} = \Big\{ f \in L^2_{\text{per}} \mid \sum_{-\infty}^{\infty} |(1 + |n|)\hat{f}(n)|^2 < \infty \Big\}, \quad \text{where } \hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} \, dx,$$

   and L^2_{per} is the set of all measurable, periodic functions on $\mathbb{R}$ with $\int_0^1 |f|^2 \, d\lambda < \infty$.
   (a) *(5 points)* Does there exist a continuous, linear operator $D : H^1_{\text{per}} \to L^2_{\text{per}}$ such that $Df = f'$ for all $f \in C^1_{\text{per}}$? Prove your answer.
   (b) *(5 points)* True or false: If $f \in H^1_{\text{per}}$ then there exists an *absolutely* continuous, periodic function g such that $f = g$ almost everywhere. Prove your answer.
7. *(10 points)* Let $p \in [1, \infty)$, and recall $L^{p,\infty}$ (the weak L^p space) is defined by

$$L^{p,\infty} = \{ f \mid \|f\|_{L^{p,\infty}} < \infty \}, \quad \text{where} \quad \|f\|_{L^{p,\infty}} = \sup_{\alpha > 0} \alpha \big(\mu\{|f| > \alpha\} \big)^{1/p}.$$

   (The function f is implicitly assumed to be measurable.)
   For which $p \in [1, \infty)$ does there exists a constant c (depending on p) such that for all $f \in L^{p,\infty}$ we have

$$\sup_{\mu(E) < \infty} \frac{1}{\mu(E)^{1 - \frac{1}{p}}} \int_E |f| \, d\mu \leqslant c \|f\|_{L^{p,\infty}}.$$

If you've completed the remainder of this exam and have time to spare, here are some fun questions. These are for your entertainment only, and *will not influence your grade.*

8. In Question 3 show additionally that we must have $p \leqslant 2$.
9. If $f \in \mathcal{S}(\mathbb{R})$, then show $\sum_{-\infty}^{\infty} f(n) = \sum_{-\infty}^{\infty} \hat{f}(n)$.