372 PDE: Midterm 2.

Wed 4/2/2014

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 50 mins. The exam has a total of 4 questions and 20 points.*
- *You may use any result from class or homework* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*
- *Difficulty wise: I expect Q1 $\leqslant$ Q2 $\leqslant$ Q3 $\leqslant$ Q4.*

$\boxed{5}$ 1. Let f be a continuous 2π periodic function, and define $g(x) = e^{ix} f(x)$. Let (c_n) be the complex Fourier coefficients of f and (d_n) be the complex Fourier coefficients of g. Given that $c_n = (1 + |n|)^{-372}$ for all $n \in \mathbb{Z}$, find d_n explicitly.

$\boxed{5}$ 2. Consider the PDE $\partial_t u + 2\partial_x u - \partial_x^2 u = 0$ for $x \in (0, \pi)$, $t > 0$, with boundary conditions $u(0, t) = 0 = u(\pi, t)$. Use separation of variables to write down the general solution to this PDE as an infinite series. [You may buy the general solution of the ODE $y'' + by' + cy = 0$ for 2 points. In this case this problem will be scored out of 3 and not 5.]

$\boxed{5}$ 3. Find an explicit formula for a function u that satisfies the heat equation $\partial_t u - \frac{1}{2}\partial_x^2 u = 0$ for $x \in \mathbb{R}, t > 0$, with initial data $u(x, 0) = \operatorname{sign}(x)$ and boundary conditions

$$\lim_{x\to\infty} u(x,t) = 1, \quad \text{and} \quad \lim_{x\to-\infty} u(x,t) = -1.$$

You may leave your answer as an unsimplified integral, provided the integrand is an explicit function.

$\boxed{5}$ 4. Let $L, T > 0$ and a, c be two functions such that $a(x,t) \geqslant 0$ and $c(x,t) \geqslant 0$. Suppose u is twice differentiable on the open rectangle $R \stackrel{\text{def}}{=} (0, L) \times (0, T)$ and satisfies the partial differential inequality

$$\partial_t u - a\partial_x^2 u + cu \leqslant 0.$$

Suppose further u is continuous on the closed rectangle $\bar{R} \stackrel{\text{def}}{=} [0, L] \times [0, T]$. If M is the maximum of u in $\bar{R}$, and $M \geqslant 0$, then show that u attains the value M on the sides or bottom of $\bar{R}$. [A more general version of this question was on Homework 4. Please provide a complete proof here without quoting and using the result from homework 4.]