372 PDE: Midterm 1.

Mon 02/10/2014

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 50 mins. The exam has a total of 4 questions and 20 points.*
- *You may use any result from class or homework* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*
- *Difficulty wise: I expect* $Q1 \leqslant Q2 \approx Q3 \ll Q4$.

[5] 1. Find the general solution of the PDE

$$\partial_x u + x \partial_y u = x^2 + y.$$

[5] 2. Suppose u satisfies the PDE

$$\partial_t^2 u + 3\partial_x \partial_t u - 2\partial_x^2 u = 0,$$

and as $x \to \pm\infty$ we have $u \to 0$, $\partial_t u \to 0$ and $\partial_x u \to 0$. Find a constant $\alpha \in \mathbb{R}$ so that the energy E defined by

$$E(t) = \int_{-\infty}^{\infty} \left[(\partial_t u(x,t))^2 + \alpha (\partial_x u(x,t))^2 \right] dx$$

is constant as a function of time.

[5] 3. Let $u(x,t)$ be the population of a virus at the point $x \in \mathbb{R}^3$ and time t. Suppose the virus population changes as follows:

(i) Due to overcrowding, the virus migrates from regions of high population to regions of low population at a rate proportional to the gradient. Namely, the rate of migration in a particular direction v equals $\kappa (\nabla u) \cdot v$, where $\kappa > 0$ is some constant.

(ii) The rate at which the virus population grows (due to reproduction and death) equals $u(1-u)$.

Find a PDE satisfied by the function u. [For *half credit*, you may instead do the *one dimensional* version of this question assuming $x \in \mathbb{R}$.]

[5] 4. Let $D \subseteq \mathbb{R}^3$ be a sphere of radius 1. We claim that there exists *only one* real number $\alpha \in \mathbb{R}$ so that the PDE

$$-\triangle u = 1 \quad \text{in the domain } D \text{ with Neumann boundary conditions} \quad \hat{n} \cdot \nabla u = \alpha \quad \text{on } \partial D$$

has a solution. Find α.