Assignment 12: Assigned Wed 04/23. Due Wed 04/30

1. **Hopf lemma** Here’s an outline to solve the optional challenge from HW12.
   
   (a) Given $0 < R_0 < R_1$, let $A(R_0, R_1)$ be the annulus $\{x \in \mathbb{R}^2 \mid R_0 < |x| < R_1\}$. Let $c_0, c_1 \in \mathbb{R}$ with $c_0 < c_1$, and suppose $v$ satisfies the PDE $-\Delta v = 0$ in $A(R_0, R_1)$, with $v = c_0$ on the inner boundary, and $v = c_1$ on the outer boundary. Show that $\partial_r v(R_1, \theta) > 0$. [Hint: Find the solution explicitly.]

   (b) Let $B_R = \{x \in \mathbb{R}^2 \mid |x| < R\}$. Suppose $u$ is some function such that $-\Delta u \leq 0$ in $B_R$, and $u$ attains a maximum at some point $x_0 \in \partial B_R$. Suppose further $u(0) < u(x_0)$. Show that $\partial_r u(x_0) > 0$. [Hint: Observe first that for some $R_0$ small enough, $c_0 = \max_{|x| = R_0} u < u(x_0)$. Let $c_1 = u(x_0)$ and use the maximum principle and previous subpart.]

   (c) **Hopf lemma** Suppose $D \subseteq \mathbb{R}^2$ is a domain with a smooth boundary. Suppose $u$ is a non-constant function satisfying $-\Delta u \leq 0$ in $D$, and is continuous up to the boundary of $D$. If $u$ attains it’s maximum at a point $x_0 \in \partial D$, show that $\frac{\partial u}{\partial n} > 0$ at $x_0$, where $\hat{n}$ is the outward pointing unit normal vector. [I stated this in 2D for simplicity; with minor modifications to part (a), your proof will work essentially unchanged in 3D as well.]

2. Sec. 7.1. 6.

3. Sec. 7.3. 2.

4. Sec. 7.4. 12, 21, 22.