21-372 PDE: Midterm 1.

April 1st, 2013

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 50 mins. The exam has a total of 3 questions and 20 points.*
- *You may use any result from class or homework* ***PROVIDED*** *it was proved* independently *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*
- *Difficulty wise: Q1 $\approx$ Q2(a) $<$ Q2(b) $<$ Q2(c) $\ll$ Q3.*

[5] 1. The solution to the PDE $\partial_t u - \frac{1}{2}\partial_x^2 u = 0$ for $x \in (0, \infty)$, $t > 0$ with initial data $u(x, 0) = f(x)$ and Neumann boundary conditions $\partial_x u(0, t) = 0$ is given by the formula

$$u(x,t) = \int_0^\infty f(y)[G(x-y,t) + G(x+y,t)]\,dy, \quad \text{where } G(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}.$$

Using this, find a formula for the solution to the PDE $\partial_t u - \frac{1}{2}\partial_x^2 u = g$ for $x \in (0, \infty)$, $t > 0$ with initial data $u(x, 0) = f(x)$ and Neumann boundary conditions $\partial_x u(0, t) = 0$.

2. Let f be a 2π periodic function such that that f is differentiable and f' is continuous. Let c_n be the complex Fourier coefficients of f, and d_n be the complex Fourier coefficients of f'.

[1] (a) State (with out proof) a relation between the c_n's and $\int_{-\pi}^{\pi} |f|^2$.

[4] (b) Find and prove a relation between c_n and d_n. [This was a question on your homework. In keeping with the exam instructions, please provide a self contained proof here.]

[5] (c) Find a number $a > 0$ such that

$$\int_{-\pi}^{\pi} |f|^2 \leqslant a \int_{-\pi}^{\pi} |f'|^2,$$

for all differentiable, 2π-periodic functions f such that f' is continuous and $\int_{-\pi}^{\pi} f(x)\,dx = 0$. Prove your answer. [Hint: This is part (c) of a question! This is also VERY DIFFERENT from the Raleigh quotient question on the homework.]

[5] 3. Let f be a function such that $\int_0^1 |f|^2 < \infty$. Suppose u satisfies the heat equation $\partial_t u - \partial_x^2 u = 0$ for $x \in (0, 1)$, $t > 0$ with initial data $u(x, 0) = f(x)$ and Dirichlet boundary conditions $u(0, t) = u(1, t) = 0$. Let $M(t) = \max\{|u(x, t)| \mid x \in [0, 1]\}$. Does $\lim_{t\to\infty} M(t)$ exist? If yes compute its value. Prove your answer. [Hint: Use separation of variables helps. Partial credit: I'll give you 3 points for a correct solution assuming $\sum |A_n| < \infty$, where A_n's are the appropriate Fourier coefficients of f. Alternately, I'll give you 4 points for a correct solution if you instead assume f is bounded.]