21-372 PDE: Final.

May 9^{th}, 2013

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 3 hours. The exam has a total of 7 questions and 70 points.*
- *You may use any result from class or homework* ***PROVIDED*** *it was proved* independently *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*
- *About half the questions are adapted from your homework. So that you don't inadvertently quote the same result from the homework (and get zero credit, by the previous rule) I have explicitly stated which homework the question was taken from.*
- *Difficulty wise: The first question is slightly easier than the rest. The last question is impossible.*

[10] 1. Find the general solution of the PDE $(\partial_x - \partial_y)^2 u = 0$. [HINT: Recall with operator notation $(\partial_x - \partial_y)^2 u = \partial_x^2 u - 2\partial_x\partial_y u + \partial_y^2 u$. This is a problem from homework #3 with the numbers changed.]

[10] 2. Suppose u satisfies the PDE $\partial_t u + 6u\partial_x u + \partial_x^3 u = 0$ for $x \in \mathbb{R}$ and $t > 0$. Suppose further that the function u and all its derivatives (including higher order derivatives) decay to 0 as $x \to \pm\infty$. Given $\alpha \in \mathbb{R}$ define

$$F(t) = \int_{-\infty}^{\infty} \Big[u(x,t)^3 - \alpha(\partial_x u)^2 \Big] \, dx.$$

Find α so that $\frac{dF}{dt} = 0$. [This was essentially problem 5 on homework #3.]

[10] 3. Suppose u satisfies

the PDE	$\partial_t^2 u - \partial_x^2 u = \sin(\pi x)$	for $x \in (0,1)$, $t > 0$,
with boundary conditions	$u(0,t) = u(1,t) = 0$	for $t > 0$,
and initial data	$u(x,0) = 3\sin(\pi x) + 7\sin(2\pi x)$ and $\partial_t u(x,0) = 0$.	

Find u. [If you can't do this, I will award half credit if you instead assume $\partial_t^2 u - \partial_x^2 u = 0$.]

4. Let f, g be two functions defined on the interval $(0,\pi)$, such that $\|f\|, \|g\| < \infty$. Let B_n be the n^{th} Fourier sine coefficient of f, and C_n be the n^{th} Fourier sine coefficient of g.

[5] (a) Compute

$$\int_0^\pi f(x)g(x)\, dx$$

in terms of the B_n's and C_n's. (No justification is required. Just guess the final answer.)

[5] (b) Rigorously prove your answer in the previous part. [If your answer involves interchanging an infinite sum and an integral, odds are it's not rigorous and will get **ZERO CREDIT**. Hint – We've proved Parseval's identity in class, which you may freely use.]

[10] 5. Let $R > 0$ and $D = \{x \in \mathbb{R}^3 \mid |x| > R\}$. Find the Greens function of D. [This is Q7.4.12 from the book, and was assigned on homework #14. Recall that the Newton potential in 3D is $N(x) = -1/(4\pi|x|)$.]

[10] 6. Let $R > 0$ and $B = \{x \in \mathbb{R}^2 \mid |x| < R\}$. Suppose u is a function such that $\triangle u = 0$ in B. Define

$$a = \int_B u = \iint_B u(x,y)\, dx\, dy.$$

Find a formula expressing $u(0,0)$ in terms of a and R.

[10] 7. *(WARNING: This is short, but TRICKY.)* Suppose $\triangle u = 0$ in *all of* $\mathbb{R}^2$, and u is bounded, then show that u must be constant. [This is called the Louiville Theorem. Hint – cleverly use the previous question.]