Math 880 Advanced Stochastic Calculus.

1 Course Description

This is an advanced Stochastic Calculus course aimed at Mathematics Ph.D. students. This course will present a rigorous, no holds barred, treatment of all the technicalities involved, and *will expect the same level of work from the students*. Undergraduates taking this course will be held to the *same* standard as graduate students.

2 Prerequisites

- Graduate level Measure theory. If you haven't taken 21-720, you should at least be well versed with Caratheodory extension, L^p spaces and the Radon Nykodim theorem.
- Graduate level Probability. If you haven't taken 21-721, then you should at least be well versed with tightness, convergence of random variables, and fundamental results for discrete time martingales (e.g. Doob's inequalities, convergence theorems, and optional sampling).

Most importantly, you should be able to fill in any holes in your background yourself as the semester progresses.

3 Tentative Syllabus

ten-ta-tive *(advective)* /ˈten-tə-tiv/. Not certain or fixed; provisional.

Note: Proofs of items marked with $\star$ are more than a simple exercise, and may be skipped in this course. Usually these are skipped because they will take too much time, or because the proof is not "illuminating". I will, however, occasionally need these results to overcome technicalities, and I encourage you to look up the proofs yourselves.

Items marked with a $\star\star$ may be skipped entirely in class. I will not use these statements through the course, though will occasionally make a reference to them when appropriate. These references, however, will mainly be to emphasize interesting connections, and I will ensure the core course material without any $\star\star$ items is completely self contained. The $\star\star$ items, however, are of fundamental importance, and I strongly recommend you familiarize yourself with them before taking the qualifying exams.

1. Continuous time Martingales. *(2.5 weeks)*
   (a) Stochastic processes, Filterations, etc.
   (b) Doob's Martingale inequalities and convergence theorems.
   (c) Optional sampling theorem and Localization.
   $\star$ (d) Existence of RCLL modifications.
   $\star\star$ (e) The Doob-Meyer decomposition.
   (f) Quadratic and joint quadratic variation.
2. Brownian motion. *(3.5 weeks)*
   (a) Daniel-Kolmogorov and Kolmogorov-Čentsov theorems.
   (b) Construction of Brownian motion via Kolmogorov-Čentsov.
   (c) Wiener space, Brownian families.
   (d) Markov and strong Markov properties.
   (e) Reflection principle, and passage time densities.
   (f) Blumenthal, Kolmogorov 0-1 laws.
   (g) Running maximum and reflected Brownian motion
   (h) Elementary sample path properties.
   (i) The law of iterated logarithm.
   (j) Weak convergence and Donsker's invariance principle.
   $\star\star$ (k) Skorohod embedding
3. Stochastic integration. *(3 weeks)*
   (a) Construction of the Itô integral with respect to local martingales.
   (b) Itô's formula.
   (c) Lévy's characterization of Brownian motion.
   $\star\star$ (d) The Burkholder-Davis-Gundy inequalities.
   (e) Martingale representation theorem.
4. The Girsanov Theorem. *(1 week)*
   (a) Statement and proof.
   (b) Regularity of exponential martingales
   $\star\star$ (c) Novikov and Kazamaki's condition.
5. Stochastic differential equations. *(1 week)*
   (a) Strong solutions. Existence and uniqueness.
   (b) Weak solutions. Existence and uniqueness.
   (c) Ornstein-Uhlenbeck and Bessel processes.
   (d) Brownian bridge.
6. Diffusions. *(2.5 weeks)*
   (a) Markov and strong Markov properties.
   (b) The generator and characteristic operators.
   (c) Dynkin's formula.
   (d) Feynman-Kac formula, and the Kolmogorov forward equation.
   (e) Dirichlet and Poisson problems.

References

[1] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, 2nd ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991. MR1121940 (92h:60127)

[2] B. Øksendal, *Stochastic differential equations*, 6th ed., Universitext, Springer-Verlag, Berlin, 2003. An introduction with applications. MR2001996 (2004e:60102)

[3] H. Kunita, *Stochastic flows and stochastic differential equations*, Cambridge Studies in Advanced Mathematics, vol. 24, Cambridge University Press, Cambridge, 1997. Reprint of the 1990 original. MR1472487 (98e:60096)

[4] A. Friedman, *Stochastic differential equations and applications*, Dover Publications Inc., Mineola, NY, 2006. Two volumes bound as one; Reprint of the 1975 and 1976 original published in two volumes. MR2295424 (2007k:60005)

[5] L. C. G. Rogers and D. Williams, *Diffusions, Markov processes, and martingales. Vol. 1*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2000. Foundations; Reprint of the second (1994) edition. MR1796539 (2001g:60188)

[6] L. C. G. Rogers and D. Williams, *Diffusions, Markov processes, and martingales. Vol. 2*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2000. Itô calculus; Reprint of the second (1994) edition. MR1780932 (2001g:60189)