880 Stochastic Calculus: Midterm.
Oct 17th

- This is a closed book test. No calculators or computational aids are allowed.
- You have 80 mins. The exam has a total of 4 questions and 20 points.
- You may use without proof any result that has been proved in class or on the homework, unless you are explicitly instructed otherwise. You must also CLEARLY state the result you want to use.

In this exam, Ω always denotes a probability space, with measure P. Brownian motion will usually be denoted by W or B, and the underlying filtration (if not explicitly mentioned) is denoted by $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$, and is always assumed to satisfy the usual conditions.

1. Let $\text{sign}^+(x) = 1$ if $x \geq 0$, and 0 if $x < 0$. Let $Y_t = \int_0^t \text{sign}^+(W_s) \, dW_s$. Find $P(Y_1 > 2)$. [You may leave your answer as an unsimplified deterministic integral, provided you have explicitly computed the integrand.]

2. Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}$. Define the random variable $X : \Omega \to \mathbb{R}$ by

$$X(\omega) \overset{\text{def}}{=} \lambda\{t \geq 0 \mid W_t(\omega) = 0\}.$$  

Compute the distribution of $X$.

3. Let $X \overset{\text{def}}{=} \{X_t, \mathcal{F}_t \mid 0 \leq t < \infty\}$ be a non-negative, right continuous super-martingale. Show $X_\infty \overset{\text{def}}{=} \lim_{t \to \infty} X_t$ exists almost surely. Must $X_\infty$ be integrable? Must $E(X_\infty \mid \mathcal{F}_t) \leq X_t$? Prove or find a counter example. [This was a question on HW#1. Please provide a complete solution here, and don’t simply say “done on homework”.]

4. Let $W$ be a 3D Brownian motion starting at 0, $e_1 = (1, 0, 0) \in \mathbb{R}^3$, and $N : \mathbb{R}^3 - \{0\} \to \mathbb{R}$ be defined by $N(x) = \frac{1}{|x|}$. Is the process $X_t$ defined by $N(W_t + e_1)$ a local martingale? Is it a martingale? Prove or disprove. [Note: Let $\tau_n = \inf\{t \geq 0 \mid |W_t + e_1| = 1/n\}$. You may assume (without proof) that $\tau_n \to \infty$ almost surely.]