1. Let $W$ be a standard 1D Brownian motion. Compute $\int_0^t \sin(W_s) \, dW_s$, and express your answer in the form $\int_0^t f(W_s, s) \, ds + g(W_t, t)$ for some explicit, deterministic functions $f$ and $g$.

Solution. By Itô’s formula,

$$d \cos(W_t) = -\sin(W_t) \, dW_t - \frac{1}{2} \cos(W_t) \, dt.$$ 

Consequently,

$$\int_0^t \sin(W_t) \, dW_t = 1 - \cos(W_t) - \int_0^t \cos(W_s) \, ds.$$

2. Let $W$ be a standard 1D Brownian motion, and $f, g \in \mathcal{L}(W)$. Define the process $X$ by

$$X_t = \int_0^t f_s \, ds + \int_0^t g_s \, dW_s.$$

Find a necessary and sufficient criterion for $f$ and $g$ which guarantees that $X$ is a standard 1D Brownian motion. Prove your answer.

Solution. We claim $X$ is a standard Brownian motion if and only if $f = 0$, and $g^2 = 1$. To see this suppose first $X$ is a Brownian motion. Then $X \in \mathcal{M}_2^2$. Since $g \in \mathcal{L}(W)$, we also have $I(g, W) \in \mathcal{M}_2^2$. Thus the process $Y$, defined by $Y_t = \int_0^t f_s \, ds$, is also in $\mathcal{M}_2^2$. Since $Y$ is also of bounded variation, we must have $Y = 0$, and hence $f = 0$. Finally computing the quadratic variation, we see

$$1 = \frac{d}{dt}(X_t) = g_t^2.$$

Conversely, if $f = 0$, then $X \in \mathcal{M}_2^2$. If $g^2 = 1$, then $\frac{d}{dt}(X_t) = g_t^2 = 1$, and so by Lévy’s criterion, $X$ is a Brownian motion.

3. Let $W$ be a two dimensional Brownian motion starting from the point $(1,0) \in \mathbb{R}^2$. Let $X_t = \ln |W_t|$, $\tau_n = \inf\{t \geq 0 \mid |W_t| \leq 1/n\}$ be the hitting time of $W$ to the closed ball $B(0,1/n)$ [remember that $W$ starts from $(1,0)$, outside this ball]. For all $n \in \mathbb{N}$, is $X_{\tau_n}$ a continuous martingale? Prove it. [Recall $X_{\tau_n}$ is the stopped process defined by $X_{\tau_n} = X_{\tau_n \wedge t}$. Further, when addressing whether $X_{\tau_n}$ or not, the filtration in question is always the (augmented) Brownian filtration.]

Solution. Let $f_n = \ln |x|$ for $|x| > 1/n$, and be some $C^2$ extension inside $B_{1/n}$. By Itô’s formula,

$$f_n(W_t) - f_n(W_0) = \sum_i \int_0^t \partial_i f_n(W_s) \, dW_s^{(i)} + \frac{1}{2} \int_0^t \triangle f_n(W_s) \, ds,$$ 

almost surely. Replacing $t$ by $\tau_n \wedge t$, and noting that $f = \ln |x|$ outside $B_{1/n}$ we see

$$X_t^{\tau_n} = \sum_i \int_0^{\tau_n \wedge t} \frac{W_s^{(i)}}{|W_s|^2} \, dW_s^{(i)} + 0 = \sum_i \int_0^t \frac{W_s^{(i)}}{|W_s|^2} \, dW_s^{(i)}$$

since $\Delta \ln |x| = 0$ in two dimensions. Thus $X_{\tau_n}$ is a continuous local martingale. Further,

$$E \int_0^t \left( \frac{W_s^{(i)}}{|W_s|^2} \right)^2 \, d(W_{\tau_n})_s \leq E \int_0^t \left( \frac{W_s^{(i)}}{|W_s|^2} \right)^2 \chi_{\{s \leq \tau_n\}} \, ds \leq E \int_0^t n^2 \, ds,$$

which is finite. Thus $X_{\tau_n}$ is a continuous martingale.

4. Let $x \in \mathbb{R}^2$ be non-zero, and $W$ be a two dimensional Brownian motion starting from $x$.

(a) Let $R > |x|$ be fixed. For every $n \in \mathbb{N}$ define

$$\tau_{n,R} = \inf\{t \geq 0 \mid W_t \notin B_R - B_{1/n}\} = \inf\{t \geq 0 \mid |W_t| \notin (1/n, R)\}.$$

Compute $\lim_{n \to \infty} P(\{|W_{\tau_{n,R}}| = R\})$.

Solution. For simplicity, let $\tau_n = \tau_{n,R}$. Let $p_n = P(\{|W_{\tau_n}| = 1/n\}$, and $q_n = 1 - p_n = P(\{|W_{\tau_n}| = R\}$). Recall $\Delta \ln |x| = 0$, and so by the Dynkin formula,

$$E^x \int_0^{\tau_n} 0 \, ds + \ln |x| = E^x \ln |W_{\tau_n}| = p_n \ln \left( \frac{1}{n} \right) + q_n \ln R.$$

Since $(\ln(1/n)) \to -\infty$, and $0 \leq q_n \leq 1$, this forces $(p_n) \to 0$. Hence

$$\lim_{n \to \infty} P(\{|W_{\tau_{n,R}}| = R\}) = \lim_{n \to \infty} (1 - p_n) = 1.$$

(b) Let $\tau = \inf\{t \mid W_t = 0\}$. Compute $P(\tau = \infty)$.

Solution. Let $\tau_R = \inf\{t \mid W_t \notin B_R - \{0\}\}$. Then $\tau_{n,R} \to \tau_R$ monotonically as $n \to \infty$. Consequently $|W_{\tau_R}| = R$ almost surely.

By continuity of paths, note that if $\tau < \infty$, then there must exist some finite (integer) $R$ such that $W_{\tau_R} = 0$. Since this happens with probability 0 for all $R$, $P(\tau < \infty) = 0$, and hence $P(\tau = \infty) = 1$.

Remark. We’ve seen in class that a 1D Brownian motion starting from $x \neq 0$, will eventually hit 0 almost surely. The content of Question [8] shows that this is false for dimension 2. Two dimensional Brownian motion starting at any point away from the origin will (almost surely) take infinite time to hit the origin. This is also the case in higher dimensions, and the only modification needed to the above proof is to replace $\ln(x)$ with the Newton potential $|x|^{-d-2}$.

5. Let $W$, $X$ be as in Question [8]. Is $X$ a martingale? Prove it. [Again, the filtration in question is the (augmented) Brownian filtration.]
Questions 3 and 4(b) combined immediately shows that $X$ is a continuous local martingale. However, $X$ is not a martingale, even though it is integrable! To prove this it is enough to show that $EX_t$ is not constant in time. Let $c_1 = (1,0)$. Then
\[ EX_t = E \ln |W_t| = \int \ln |x| e^{-|x-c_1|^2/2t} \frac{dx}{2\pi t} = \int \ln \left| e_1 + y\sqrt{t} e^{-|y|^2/2t} \frac{dy}{2\pi} \right|, \]
and so $\lim_{t \to \infty} EX_t = \infty$. However $EX_0 = E \ln 1 = 0$. \hfill \Box

6. Let $M$ be a continuous martingale, and suppose $EM_t^2 \leq c(1 + t^2)$ for some $c, \beta > 0$. For any $\alpha > \frac{\beta}{2}$ show that $\lim_{t \to \infty} \frac{|M_t|}{t^{\alpha}}$ exists almost surely, and compute it's value.

**Solution.** For any $\varepsilon > 0$, we have
\[ P \left[ \sup_{2^n \leq t \leq 2^{n+1}} \frac{|M_t|}{t^{\alpha}} \geq \varepsilon \right] \leq P \left[ \sup_{2^n \leq t \leq 2^{n+1}} \frac{|M_t|^2}{2^{2n\alpha}} \geq \varepsilon^2 \right] \leq \frac{c(1 + 2^{\beta(n+1)})}{2^{2n\alpha} \varepsilon^2}, \]
which is summable for $\alpha > \frac{\beta}{2}$. Thus by the Borel Cantelli lemma, these events can't occur infinitely often. That is, for almost every $\omega \in \Omega$, there exists $N(\omega)$ such that for $n > N(\omega)$
\[ \sup_{2^n \leq t \leq 2^{n+1}} \frac{|M_t(\omega)|}{t^{\alpha}} < \varepsilon. \]
Thus $\lim_{t \to \infty} |M_t|/t^{\alpha} = 0$ almost surely. \hfill \Box

7. Let $X$ be a $d$-dimensional diffusion satisfying the SDE
\[ dX_t = b(X_t) dt + \sigma(X_t) dW_t, \]
where $b$ and $\sigma$ are time independent and Lipschitz. Let $D \subseteq \mathbb{R}^d$ be a domain, and $\tau$ be the exit time of $X$ from $D$. Suppose
\[ u \in C^{2,1}(D \times (0, \infty)) \cap C(\bar{D} \times (0, \infty)) \cap C(D \times [0, \infty)) \]
satisfies
\[ \partial_t u - Lu = 0 \quad \text{in } D, \quad u(x, 0) = 1 \quad \text{in } D, \quad u(x, t) = 0 \quad \text{on } \partial D \times (0, \infty), \]
where $L = \sum_i b_i \partial_i + \frac{1}{2} \sum_{i,j} a_{i,j} \partial_i \partial_j$, and $a_{i,j} = \sum_k \sigma_{i,k} \sigma_{j,k}$. Show that $u(x,t) = P^x(\tau \geq t)$. [Note: The maximum principle guarantees $0 \leq u(x,t) \leq 1$, which you may use in your solution.]

**Solution.** Fix $T > 0$, and put $Y = X^\tau$, and $Z_t = u(Y_t, T - (\tau \wedge t))$. By Itô's formula,
\[ dZ_t = -\partial_t u(Y_t, T - (\tau \wedge t)) d(\tau \wedge t) + \sum_i \partial_i u(Z_t, T - t) dY_t^{(i)} + \frac{1}{2} \sum_{i,j} \partial_i \partial_j u(Y_t, T - t) d(Y_t^{(i)}, Y_t^{(j)})_t \]
This simplifies to
\[ dZ_t = (-\partial_t + L) u(Y_t, T - (\tau \wedge t)) d(\tau \wedge t) + \sum_{i,j} \partial_i u(Y_t, T - t) \sigma_{i,j} dW_t^{(j)} \]
\[ = \sum_{i,j} \partial_i u(Y_t, T - \tau \wedge t) \sigma_{i,j} dW_t^{(j)} \]
because $(-\partial_t + L) u = 0$. Thus $Z$ is a continuous local martingale. By the maximum principle, $0 \leq u \leq 1$, and hence $0 \leq Z \leq 1$. This immediately implies $Z$ is a continuous martingale. Thus taking expectations, and sending $t \to T^-$, we have $E^x Z_0 = E^x Z_T$. Of course,
\[ E^x Z_T = E^x u(x_0, T) = u(x, T). \]
Using the boundary conditions for $u$,
\[ E^x Z_0 = E^x u(x_{\tau \wedge T}, T - \tau \wedge T) \]
\[ = E^x \left( \chi_{(\tau < T)} u(x_\tau, T - \tau) + \chi_{(\tau \geq T)} u(x_T, 0) \right) \]
\[ = E^x \left( \chi_{(\tau < T)} 0 + \chi_{(\tau \geq T)} \right) = P^x(\tau \geq T). \]
and hence $u(x, T) = P^x(\tau \geq T)$ as desired. \hfill \Box