Math 720: Homework.

Do, but don't turn in optional problems. *There is a firm 'no late homework' policy.*

Assignment 1: Assigned Wed 09/05. Due Wed 09/12

Following the notation of Cohn, I use λ to denote the Lebesgue measure.

1. For each of the following sets, compute the Lebesgue outer measure.
   (a) Any countable set. (b) The Cantor set. (c) $\{x \in [0,1] \mid x \notin \mathbb{Q}\}$.
2. (a) If $V \subseteq \mathbb{R}^d$ is a subspace with $\dim(V) < d$, then show that $\lambda(V) = 0$.
   (b) If $P \subseteq \mathbb{R}^2$ is a polygon show that $\operatorname{area}(P) = \lambda(P)$.
3. (a) Say μ is a *translation invariant* measure on $(\mathbb{R}^d, \mathcal{L})$ (i.e. $\mu(x + A) = \mu(A)$ for all $A \in \mathcal{L}$, $x \in \mathbb{R}^d$) which is finite on bounded sets. Show that $\exists c \geqslant 0$ such that $\mu(A) = c\lambda(A)$.
   (b) Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an orthogonal linear transformation, and $A \in \mathcal{L}$. Show that $T(A) \in \mathcal{L}$ and $\lambda(T(A)) = \lambda(A)$. [Hint: Express T in terms of elementary transformations.]
4. (a) Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset \in \mathcal{E}$, $X \in \mathcal{E}$ and $\rho(\emptyset) = 0$. For any $A \subseteq X$ define
$$\mu^*(A) = \inf\Big\{\sum_1^\infty \rho(E_i) \Bigm| E_i \in \mathcal{E}, \text{ and } A \subseteq \bigcup_1^\infty E_j\Big\}.$$
   Show that μ^* is an outer measure.
   (b) Let (X, d) be any metric space, $\delta > 0$ and define $\mathcal{E}_\delta = \{B(x, r) \mid x \in X, r \in (0, \delta)\}$. Given $\alpha > 0$ define $\rho(B(x, r)) = c_\alpha r^\alpha$, where $c_\alpha = \pi^{\alpha/2}/\Gamma(1+\alpha/2)$ is a normalization constant. Let $H^*_{\alpha,\delta}$ be the outer measure obtained with this choice of ρ and the collection of sets $\mathcal{E}_\delta$. Define $H^*_\alpha = \lim_{\delta\to 0} H^*_{\alpha,\delta}$. Show H^*_α is an outer measure and restricts to a measure H_α on a σ-algebra that contains all Borel sets. The measure H_α is called the *Hausdorff measure of dimension α*. [Don't reprove Caratheodory.]
   (c) If $X = \mathbb{R}^d$, and $\alpha = d$ show that H_d is the Lebesgue measure.
   (d) Let $S \in \mathcal{B}(X)$. Show that there exists (a unique) $d \in [0, \infty]$ such that $H_\alpha(S) = \infty$ for all $\alpha \in (0, d)$, and $H_\alpha(S) = 0$ for all $\alpha \in (d, \infty)$. This number is called the *Hausdorff dimension* of the set S.
   (e) Compute the Hausdorff dimension of the Cantor set.

Details in class I left for you to check. (Do it, but don't turn it in.)

* We saw in class $\ell(I) = I$ for closed cells. Show it for arbitrary cells.
* Show that $m^*(a + E) = m^*(E)$ for all $a \in \mathbb{R}^d$, $E \subseteq \mathbb{R}^d$.
* Show that the arbitrary intersection of σ-algebras on X is also a σ-algebra.
* Verify that the counting measures and delta measures are measures.
* When proving Caratheodory, we proved in class Σ is a σ-algebra, and that $\mu^*|_\Sigma$ is *finitely* additive. Show that $\mu^*|_\Sigma$ is countably additive.

Assignment 2: Assigned Wed 09/12. Due Wed 09/19

1. Let (X, Σ, μ) be a measure space. For $A \in P(X)$ define $\mu^*(A) = \inf\{\mu(E) \mid E \supseteq A \,\&\, E \in \Sigma\}$, and $\mu_*(A) = \sup\{\mu(E) \mid E \subseteq A \,\&\, E \in \Sigma\}$.
   (a) Show that μ^* is an outer measure.
   (b) Let $A_1, A_2, \dots \in \mathcal{P}(X)$ be disjoint. Show that $\mu_*(\bigcup_1^\infty A_i) \geqslant \sum_1^\infty \mu_*(A_i)$. [The set function μ_* is called an *inner measure*.]
   (c) Show that for all $A \subseteq X$, $\mu^*(A) + \mu_*(A^c) = \mu(X)$.
   (d) Let $A \subseteq \mathcal{P}(X)$ with $\mu^*(A) < \infty$. Show that $A \in \Sigma_\mu \iff \mu_*(A) = \mu^*(A)$.
2. Here's an alternate (cleaner) approach to proving $\mathcal{L} = \mathcal{B}_\lambda$. We do it by proving a stronger statement than necessary.
   (a) If $A \in \mathcal{L}(\mathbb{R}^d)$ show that for any $\varepsilon > 0$ there exists two sets C, U such that $C \subseteq A \subseteq U$, C is closed, U is open and $\lambda(U - C) < \varepsilon$.
   (b) For $A \in \mathcal{L}(\mathbb{R}^d)$, show that that there exists an F_σ, F and a G_δ, G such that $F \subseteq A \subseteq G$ and $\lambda(G - F) = 0$. Conclude $\mathcal{B}_\lambda = \mathcal{L}$.
3. Let $A \in \mathcal{L}(\mathbb{R}^d)$. Prove every subset of A is Lebesgue measurable $\iff \lambda(A) = 0$.
4. (a) Prove $\mathcal{B}(\mathbb{R}^{m+n}) = \sigma(\{A \times B \mid A \in \mathcal{B}(\mathbb{R}^m) \,\&\, B \in \mathcal{B}(\mathbb{R}^n)\})$.
   (b) Prove $\mathcal{L}(\mathbb{R}^{m+n}) \supsetneq \sigma(\{A \times B \mid A \in \mathcal{L}(\mathbb{R}^m) \,\&\, B \in \mathcal{L}(\mathbb{R}^n)\})$.
   (c) Show $\mathcal{L}(\mathbb{R}^2) \supsetneq \mathcal{B}(\mathbb{R}^2)$.
5. Find $E \in \mathcal{B}(\mathbb{R})$ so that for all $a < b$, we have $0 < \lambda(E \cap (a, b)) < b - a$.

We say $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra if $\emptyset \in \mathcal{A}$, and $\mathcal{A}$ is closed under complements and *finite* unions. We say $\mu_0 : \mathcal{A} \to [0, \infty]$ is a (positive) *pre-measure* on $\mathcal{A}$ if $\mu_0(\emptyset) = 0$, and for any countable disjoint sequence of sets sequence $A_i \in \mathcal{A}$ *such that* $\bigcup_1^\infty A_i \in \mathcal{A}$, we have $\mu_0(\bigcup_1^\infty A_i) = \sum_1^\infty \mu_0(A_i)$.

Namely, a pre-measure is a finitely additive measure on an algebra $\mathcal{A}$, which is also countably additive for disjoint unions *that belong to the algebra.*

6. *(Caratheodory extension)* If $\mathcal{A}$ is an algebra, and μ_0 is a pre-measure on $\mathcal{A}$, show that there exists a measure μ defined on $\sigma(\mathcal{A})$ that extends μ_0.

Optional problems, and details in class I left for you to check.

* Prove any open subset of $\mathbb{R}^d$ is a countable union of cells. Conclude $\mathcal{L} \supseteq \mathcal{B}$.
* Show that the cardinality $\mathcal{B}(\mathbb{R})$ is the same as that of $\mathbb{R}$, however, the cardinality of $\mathcal{L}(\mathbb{R})$ is the same as that of $\mathcal{P}(\mathbb{R})$. Conclude $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R})$. [There are of course other ways to prove this.]
* If $A_i \in \Sigma$ are such that $A_i \supseteq A_{i+1}$, show that $\mu(\bigcap_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$, provided $\mu(A_1) < \infty$. Given an example to show this is not true if $\mu(A_1) = \infty$.
* We saw in class $\lambda(A) = \sup\{\lambda(K) \mid K \subseteq A \,\&\, K \text{ is compact}\}$ for all bounded sets $A \in \mathcal{L}$. Prove it for arbitrary $A \in \mathcal{L}$.
* Show that there exists $A \subseteq \mathbb{R}$ such that if $B \subseteq A$ and $B \in \mathcal{L}$ then $\lambda(B) = 0$, *and further*, if $B \subseteq A^c$ and $B \in \mathcal{L}$ then $\lambda(B) = 0$.

Assignment 3: Assigned Wed 09/19. Due Wed 09/26

1. Let X be a topological space, and μ be a regular Borel measure on X. Show that X has a *maximal* open set of measure 0. Namely, show that there exists $U \subseteq X$, such that U open set, $\mu(U) = 0$ and further for any open set $V \subseteq X$ with $\mu(V) = 0$, we must have $V \subseteq U$. [The complement of U is defined to be the *support* of the measure μ, and denoted by $\operatorname{supp}(\mu)$.]
2. Let $\Sigma \supseteq \mathcal{B}(\mathbb{R}^d)$, and μ be a regular measure on $(\mathbb{R}^d, \Sigma)$. Suppose $A \in \Sigma$ is σ-finite (i.e. $A = \cup_1^\infty A_n$, and $\mu(A_n) < \infty$). Show that $\mu(A) = \sup\{\mu(K) \mid K \subseteq A \text{ is compact}\}$. [This remains true if we replace $\mathbb{R}^d$ with any Hausdorff space.]
3. Let μ, ν be two measures on (X, Σ). Suppose $\mathcal{C} \subseteq \Sigma$ is a π-system such that $\mu = \nu$ on $\mathcal{C}$.
   (a) Suppose $\exists C_i \in \mathcal{C}$ such that $\bigcup_1^\infty C_i = X$ and $\mu(C_i) = \nu(C_i) < \infty$. Show that $\mu = \nu$ on $\sigma(\mathcal{C})$.
   (b) If we drop the finiteness condition $\mu(C_i) < \infty$ is the previous subpart still true? Prove or find a counter example.
4. Let $\kappa \in (0, 1)$. Does there exist $E \in \mathcal{L}(\mathbb{R})$ such that for all $a < b \in \mathbb{R}$, we have $\kappa(b - a) \leqslant \lambda(I \cap (a, b)) \leqslant (1 - \kappa)(b - a)$? Prove or find a counter example. [I'm aware that this looks suspiciously like a homework problem you already did. Also, this problem has a short, elegant solution using only what we've seen in class so far.]
5. For $i \in \{1, 2\}$, let (X_i, Σ_i, μ_i) be two measure spaces with $\mu_i(X_i) < \infty$. Define $\Sigma_1 \otimes \Sigma_2 = \sigma\{A_1 \times A_2 \mid A_i \in \Sigma_i\}$.
   (a) Let $x_1 \in X_1$ and $A \in \Sigma_1 \otimes \Sigma_2$. Let $S_{x_1}(A) = \{x_2 \in X_2 \mid (x_1, x_2) \in A\}$, and $T_{x_2}(A) = \{x_1 \in X_1 \mid (x_1, x_2) \in A\}$. Show that $S_{x_1}(A) \in \Sigma_2$ and $T_{x_2}(A) \in \Sigma_1$.
   (b) If $A \in \mathcal{P}(X_1 \times X_2)$ is such that for all $x_i \in X_i$, $S_{x_1}(A) \in \Sigma_2$ and $S_{x_2}(A) \in \Sigma_1$. Must $A \in \Sigma_1 \otimes \Sigma_2$?
   (c) Show that there exists a measure ν on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ such that for all $A_i \in \Sigma_i$ we have $\nu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$.
6. *(An alternate approach to λ-systems.)* Let $\mathcal{M} \subseteq P(X)$. We say $\mathcal{M}$ is a *Monotone Class*, if whenever $A_i, B_i \in \mathcal{M}$ with $A_i \subseteq A_{i+1}$ and $B_i \supseteq B_{i+1}$ then $\bigcup_1^\infty A_i \in \mathcal{M}$ and $\bigcap_1^\infty B_i \in \mathcal{M}$. If $\mathcal{A} \subseteq P(X)$ is an algebra, then show that the *smallest* monotone class containing $\mathcal{A}$ is exactly $\sigma(A)$. [You should also address existence of a smallest monotone class containing $\mathcal{A}$.]

Optional problems, and details in class I left for you to check.

* Let X be a second countable locally compact Hausdorff space, and μ be a Borel measure on X that is finite on compact sets. Show that μ is regular.
* Is any σ-finite Borel measure on $\mathbb{R}^d$ regular?
* Show that any λ-system that is also a π-system is a σ-algebra.
* If Π is a π-system, then $\lambda(\Pi) = \sigma(\Pi)$. (We only proved $\lambda(\Pi) \subseteq \sigma(\Pi)$.)

Assignment 4: Assigned Wed 09/26. Due Wed 10/03

1. Let $f : X \to \mathbb{R}$ be measurable, and $g : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. True or false: $g \circ f : X \to \mathbb{R}$ is measurable? Prove or find a counter example.
2. Let (X, Σ) be a measure space, and $f, g : X \to [-\infty, \infty]$ be measurable. Suppose whenever $g = 0$, $f \neq 0$, and whenever $f = \pm\infty$, $g \in (-\infty, \infty)$. Show that $\frac{f}{g} : X \to [-\infty, \infty]$ is measurable. [Note that by the given data you will never get a 'meaningless' quotient of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. The remainder of the quotients (e.g. $\frac{1}{\infty}$) can be defined in the natural manner.]
3. Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions such that $(f_n) \to f$ almost everywhere (a.e.). Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function.
   (a) If for a.e. $x \in X$, g is continuous at $f(x)$, then show $(g \circ f_n) \to g \circ f$ a.e.
   (b) Is the previous part true without the continuity assumption on g?
4. Let $C \subseteq \mathbb{R}^d$ be convex. Must C be Lebesgue measurable? Must C be Borel measurable? Prove or find counter examples. [The cases $d = 1$ and $d > 1$ are different.]
5. Let (X, Σ, μ) be a measure space, and $(X, \Sigma_\mu, \bar{\mu})$ it's completion. Show that $g : X \to [-\infty, \infty]$ is Σ_μ-measurable if and only if there exists two Σ-measurable functions $f, h : X \to [-\infty, \infty]$ such that $f = h$ μ-almost everywhere, and $f \leqslant g \leqslant h$ everywhere.
6. Let X be a metric space, $\Sigma \supseteq \mathcal{B}(X)$ a σ-algebra on X, and μ a regular finite measure on (X, Σ). Let $f : X \to \mathbb{R}$ be measurable.
   (a) For any $\varepsilon > 0$ and $i \in \mathbb{N}$, show that there exists finitely many disjoint compact sets $\{K_{i,j} \mid |j| \leqslant N_i\}$ such that
   $$\mu\Big(X - \bigcup_{j=-N_i}^{N_i} K_{i,j}\Big) < \frac{\varepsilon}{2^i}, \quad \text{and} \quad f(K_{i,j}) \subseteq \Big[\frac{j}{2^i}, \frac{j+1}{2^i}\Big)$$
   (b) *(Lusin's Theorem)* For any $\varepsilon > 0$ show that there exists $K_\varepsilon \subseteq X$ compact such that $f : K_\varepsilon \to \mathbb{R}$ is *continuous*, and $\mu(X - K_\varepsilon) < \varepsilon$. [Hint: Let $K_\varepsilon = \bigcap_{i=1}^\infty \bigcup_{|j| \leqslant N_i} K_{i,j}$. Define $g_i : K_\varepsilon \to \mathbb{R}$ by $g_i(x) = j/2^i$ if $x \in K_{i,j}$ and $|j| \leqslant N_i$. Show $g_i : K \to \mathbb{R}$ is continuous and $(g_i) \to f$ uniformly on K_ε.]

   A standard extension theorem now shows that for any $f : X \to \mathbb{R}$ measurable and $\varepsilon > 0$, there exists $g_\varepsilon : X \to \mathbb{R}$ *continuous* such that $\mu\{f \neq g_\varepsilon\} < \varepsilon$.

Optional problems, and details in class I left for you to check.

* Show that $f : X \to [-\infty, \infty]$ is measurable if and only if any of the following conditions hold
  (a) $\{f < a\} \in \Sigma$ for all $a \in \mathbb{R}$.
  (b) $\{f > a\} \in \Sigma$ for all $a \in \mathbb{R}$.
  (c) $\{f \leqslant a\} \in \Sigma$ for all $a \in \mathbb{R}$.
  (d) $\{f \geqslant a\} \in \Sigma$ for all $a \in \mathbb{R}$.
* Let $f : [0, 1] \to [0, 1]$ be the Cantor function, and $g(x) = \inf\{f = x\}$. Show that f is continuous, and the range of g is the Cantor set. Are f, g Hölder continuous? If yes, what are the largest exponents α, β for which f, g are respectively Hölder-α and Hölder-β continuous.

Assignment 5: Assigned Wed 10/03. Due Wed 10/10

1. (a) Suppose $I \subseteq \mathbb{R}^d$ is a cell, and $f : I \to \mathbb{R}$ is Riemann integrable. Show that f is measurable, Lebesgue integrable and that the Lebesgue integral of f equals the Riemann integral.

   (b) Is the previous subpart true if we only assume that an improper (Riemann) integral of f exists? Prove or find a counter example.

2. (a) Let (X, Σ, μ) be a complete measure space, $f : X \to [-\infty, \infty]$ be measurable and suppose $\int_X f\,d\mu$ is defined. If $g : X \to [-\infty, \infty]$ is such that $f = g$ a.e., then show $\int_X f\,d\mu = \int_X g\,d\mu$.

   All the convergence theorems we've seen so far hold if we replace pointwise convergence with a.e. convergence. I ask you to prove one below; you should verify the others on your own.

   (b) Suppose (f_n) is a sequence of measurable functions, $f_n \geqslant 0$ a.e., and $(f_n) \to f$ a.e. on E. Show that $\liminf \int_E f_n\,d\mu \geqslant \int_E f\,d\mu$.

3. Let $f : \mathbb{R}^d \to [-\infty, \infty]$ be an integrable function such that $\int_I f\,d\lambda = 0$ for all cells I. Must $f = 0$ a.e.? Prove or find a counter example.

4. Let $f : [0, \infty) \to \mathbb{R}$ be a measurable function. We define the *Laplace Transform* of f to be the function $F(s) = \int_0^\infty \exp(-st) f(t)\,dt$ wherever defined.

   (a) If $\int_0^\infty |f(t)|\,dt < \infty$, show that $F : [0, \infty) \to \mathbb{R}$ is continuous.

   (b) If $\int_0^\infty t|f(t)|\,dt < \infty$, show that $F : [0, \infty) \to \mathbb{R}$ is differentiable.

   (c) If f is continuous and bounded, compute $\lim_{s\to\infty} sF(s)$.

5. (a) Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be linear, and $A \in \mathcal{L}$. Show that $\lambda(T(A)) = |\det(T)|\lambda(A)$. [Hint: Check it separately for $\det(T) = 0$. For $\det(T) \neq 0$, write T as a product of elementary transformations, and check the result for cells. (This should have been on HW1, but I 'inadvertently' added the assumption that T was orthogonal.)]

   (b) *(Linear change of variable)* Let $f : \mathbb{R}^d \to \mathbb{R}$ be integrable, $T : \mathbb{R}^d \to \mathbb{R}^d$ an invertible linear transformation, and $E \in \mathcal{L}(\mathbb{R}^d)$. Show that

$$\int_{T^{-1}(E)} (f \circ T)|\det T|\,d\lambda = \int_E f\,d\lambda.$$

Optional problems, and details in class I left for you to check.

* For simple functions, check that $\int_E s$ is well defined.
* For positive functions check $f \leqslant g \implies \int_E f \leqslant \int_E g$.
* For arbitrary integrable functions, check $\int_E \alpha f\,d\mu = \alpha \int_E f\,d\mu$.
* If $\int_X f\,d\mu < \infty$, then show $f < \infty$ a.e.
* If $\int_X |f|\,d\mu = 0$, then show that $f = 0$ a.e.
* Prove the following generalization of Fatou's Lemma: If $f_n \geqslant 0$ are measurable, then $\liminf \int_E f_n\,d\mu \geqslant \int_E \liminf f\,d\mu$.
* Finish the proof of showing $\int_X g\,d\mu = \int_Y g \circ f\,d\mu_{f^{-1}}$. Use this to give a quick proof that $\int_{\mathbb{R}^d} f(x+y)\,dx = \int_{\mathbb{R}^d} f(x)\,dx$. (This trick also helps with Q5(b).)

Assignment 6: Assigned Wed 10/10. Due Never

In view of your Midterm on 10/17, this homework is optional.

* If $\mu(E) = 0$, and $f : E \to [-\infty, \infty]$ is any measurable function, then show directly from the definition that $\int_E f\,d\mu = 0$.
* Let μ be the counting measure on $\mathbb{N}$, and $f : \mathbb{N} \to \mathbb{R}$ a function.
  (a) If $\sum_1^\infty |f(n)| < \infty$, then show that $\sum_{n=1}^\infty f(n) = \int_{\mathbb{N}} f\,d\mu$.
  (b) If the series $\sum_{n=1}^\infty f(n)$ is conditionally convergent, show that $\int_{\mathbb{N}} f\,d\mu$ is not defined.
* Let (X, Σ, μ) be a measure space and $f : X \to Y$ some function. Define $\tau = \{A \subseteq Y \mid f(f^{-1}(A)) = A, \& f^{-1}(A) \in \Sigma\}$. For $A \in \tau$, define $\mu_f(A) = \mu(f(A))$. Show that (Y, τ, μ_f) is a measure space. If $g : Y \to [-\infty, \infty]$ is integrable, can you write $\int_Y g\,d\mu_f$ in terms of an integral over X with respect to μ?
* Let $g \geqslant 0$ be measurable, and define $\nu(A) = \int_A g\,d\mu$. Show that ν is a measure, and $\int_E f\,d\nu = \int_E fg\,d\mu$.
* Let $f \sim g$ if $\mu\{f \neq g\} = 0$. For $p \in [1, \infty)$, define

$$\mathcal{L}^p = \{f : X \to \mathbb{R} \text{ measurable, such that } \int_X |f|^p\,d\mu < \infty\} \quad \text{and} \quad L^p = \mathcal{L}^p/\sim.$$

  For $f \in L^p$, pick any $f' \in f$, and define $\|f\|_p = (\int_X |f'|^p\,d\mu)^{1/p}$. Show that this is well defined and satisfies all the axioms of a Banach space except completeness and the triangle inequality. [Completeness and the triangle inequality are of course true but are harder to prove. I will prove them in class.]
* Show that $f \leqslant \operatorname{ess\,sup}_X f$ almost everywhere.
* For $p \in [0, 1)$ show that you need not have $\|f+g\|_p \leqslant \|f\|_p + \|g\|_p$.
* Prove Hölder's inequality if $p = 1$ or $p = \infty$.
* (a) Prove $\|f\|_1 = \sup_{\|g\|_\infty = 1} \int_X fg\,d\mu$.
  (b) If X is σ-finite, then show $\|f\|_\infty = \sup_{\|g\|_1 = 1} \int_X fg\,d\mu$.
* (a) *(Young's inequality)* Let $x, y \in \mathbb{R}$, $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that $|xy| \leqslant \frac{|x|^p}{p} + \frac{|y|^q}{q}$, and equality holds if and only if $|x|^p = |y|^q$.
  (b) Use Young's inequality to give an alternate proof of Hölder's inequality.
* (a) Suppose φ is a strictly convex function and $\mu(X) = 1$. For what functions can you have equality in Jensen's inequality. Namely, when is $\varphi(\int_X f\,d\mu) = \int_X \varphi \circ f\,d\mu$?
  (b) For what functions f, g can you have equality in Hölder's inequality?

Assignment 7: Assigned Wed 10/17. Due Wed 10/24

1. (a) If $\mu(X) < \infty$, $1 \leqslant p < q$, show $L^q(X) \subseteq L^p(X)$ and the inclusion map from $L^q(X) \to L^p(X)$ is continuous. Find an example where $L^q(X) \subsetneq L^p(X)$. [HINT: Show $\|f\|_p \leqslant \mu(X)^{\frac{1}{p}-\frac{1}{q}} \|f\|_q$.]
   (b) Let $\ell^p = L^p(\mathbb{N})$ with respect to the counting measure. If $1 \leqslant p < q$ show that $\ell^p \subsetneq \ell^q$. Is the inclusion map $\ell^p \hookrightarrow \ell^q$ continuous? Prove your answer.
2. (a) Suppose $p, q, r \in [1, \infty]$ with $p < q < r$. Prove that for all $f \in L^p \cap L^r$, $f \in L^q$. Further, find $\theta \in (0, 1)$ such that $\|f\|_q \leqslant \|f\|_p^\theta \|f\|_r^{1-\theta}$.
   (b) If for some $p \in [1, \infty)$, $f \in L^p(X) \cap L^\infty(X)$ show that $\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$. [This sort of justifies the notation $\|\cdot\|_\infty$.]
   (c) Let $p_0 \in (0, \infty]$, $\mu(X) = 1$ and $f \in L^{p_0}(X)$. Prove $\lim_{p\to 0^+} \|f\|_p = \exp\left(\int_X \ln|f| \, d\mu\right)$.
3. For any $p \in [1, \infty]$, show that simple functions are dense in $L^p(X)$. That is, for any $\varepsilon > 0$, $f \in L^p(X)$ show that there exists a simple function $s \in L^p(X)$ such that $\|f - s\|_p < \varepsilon$.
4. Let X be a metric space and μ be a regular Borel measure on $(X, \mathcal{B}(X))$. Assume further and $X = \bigcup_1^\infty U_n$, where U_n is open, $\overline{U}_n$ is compact, and $\overline{U}_n \subseteq U_{n+1}$.
   (a) For any $p \in [1, \infty)$, show that continuous compactly supported functions are dense in $L^p(X)$. [You may assume the *Tizete extension theorem* from topology, which guarantees (in a more general situation) that if $C \subseteq X$ is closed and $f : C \to \mathbb{R}$ is continuous, then there exists a continuous function $F : X \to \mathbb{R}$ such that $F = f$ on C.]
   (b) Is the previous part true for $p = \infty$? Prove or find a counter example.
5. (a) Suppose $p \in [1, \infty)$, and $f \in L^p(\mathbb{R}^d, \lambda)$. For $y \in \mathbb{R}^d$, let $\tau_y f : \mathbb{R}^d \to \mathbb{R}$ be defined by $\tau_y f(x) = f(x - y)$. Show that $(\tau_y f) \to f$ in L^p as $|y| \to 0$.
   (b) What happnes for $p = \infty$?

Optional problems, and details in class I left for you to check.

* If $p_i, q \in [1, \infty]$ with $\sum_1^N \frac{1}{p_i} = \frac{1}{q}$, show that $\|\prod_1^n f_i\|_q \leqslant \prod \|f_i\|_{p_i}$.
* Let $0 < p < q < \infty$. Then $L^p \not\subseteq L^q$ iff X contains sets of arbitarily small, positive, measure. Also, $L^q \not\subseteq L^p$ iff X contains sets of arbitarily large (but finite) measure.
* *(Vitali's convergence theorem.)* Let $f_n, f \in L^1$. Show that $(f_n) \to f$ in L^1 if and only if (1) $(f_n) \to f$ in measure, (2) $\{f_n\}$ is uniformly integrable, and (3) For all $\varepsilon > 0$ there exists $F \in \Sigma$ with $\mu(F) < \infty$ such that $\int_{F^c} |f_n| < \varepsilon$. [I proved the forward direction in class, and sketched the reverse. Fill in the details of the reverse.]

Assignment 8: Assigned Wed 10/24. Due Wed 10/31

1. Suppose $\Sigma = \sigma(\mathcal{C})$, where $C \subseteq \mathcal{P}(X)$ is countable. If μ is a σ-finite measure and $1 \leqslant p < \infty$, show that $L^p(X)$ is seperable (i.e. has a countable dense subset).
2. Let $e_n(x) = e^{2\pi i n x}$, $X = [0, 1]$. For what $p \in [1, \infty]$ does $\{e_n\}$ have a convergent subsequence in $L^p(X, \lambda)$? Prove it.
3. (a) Suppose $\lim_{\lambda\to\infty} \sup_n \int_{|f_n|>\lambda} |f_n| \, d\mu = 0$. Show that there exists an increasing funciton φ with $\varphi(\lambda)/\lambda \to \infty$ as $\lambda \to \infty$, such that $\sup_n \int_X \varphi(|f_n|) < \infty$.
   (b) Suppose $\{f_n\}$ is uniformly integrable, and $\sup_n \int |f_n| < \infty$. Show that $\lim_{\lambda\to\infty} \sup_n \int_{|f_n|>\lambda} |f_n| = 0$.
   (c) Show that the previous part fails without the assumption $\sup_n \int |f_n| < \infty$.
4. Recall we defined the variation of μ by $|\mu| = \mu^+ + \mu^-$, and the total variation by $\|\mu\| = |\mu|(X)$. (You should check that these are well defined.)
   (a) If μ, ν are two signed measurs on X, show that $|\mu+\nu|(A) \leqslant |\mu|(A) + |\nu|(A)$.
   (b) Let $\mathcal{M}$ be the space of all finite signed measures on (X, Σ). Show that $\mathcal{M}$ with total variation norm (i.e. with $\|\mu\| = |\mu|(X)$) is a Banach space.
   (c) Show that $(\mu_n) \to \mu$ if and only if $(\mu_n(A)) \to \mu(A)$ uniformly in A, $\forall A \in \Sigma$.
5. (a) For a signed measure, we define $\int_X f \, d\mu = \int_X f \, d\mu^+ - \int_X f \, d\mu^-$. Suppose $(f_n) \to f$, $(g_n) \to g$, and $|f_n| \leqslant g_n$ almost everywhere with respect to $|\mu|$. If $\lim \int_X g_n \, d|\mu| = \int_X g \, d|\mu| < \infty$, show that $\lim \int_X f_n \, d\mu = \int_X f \, d\mu$.
   (b) Suppose $f, f_n \in L^1$, and $(f_n) \to f$ almost everywhere. Show that $\lim \int |f_n - f| d|\mu| = 0$ if and only if $\lim \int |f_n| \, d|\mu| = \int |f| d|\mu|$.

Optional problems, and details in class I left for you to check.

* Show $L^\infty(\mathbb{R})$ is not separable.
* Say μ is a signed measure, and $A_i \in \Sigma$ are pariwise disjoint. If $|\mu(\bigcup A_i)| < \infty$, then must $\sum_1^\infty |\mu(A_i)| < \infty$? Prove, or find a counter example.
* If $g \in L^1(X, \mu)$, let $\nu(A) = \int_A g$. Show that ν is a signed measure on X, and $\int f \, d\nu = \int f g \, d\mu$.
* (a) Prove the Hanh decomposition is unique, up to sets of measure 0. [That is show $X = P_1 \cup N_1$ and $X = P_2 \cup N_2$, then $P_2 = P_1 - A \cup B$, where all subsets of A, B have measure 0, and a similar statement for N.]
  (b) Show that the measures μ^+ and μ^- we defined in class are independent of the Hanh decomposition used to define them.
  (c) We say μ and ν are *mutually singular* if $X = A \cup B$ where $A, B \in \Sigma$ with $A \cap B = \emptyset$, and for all measurable $A' \subseteq A$, $B' \subseteq B$ we have $\mu(A') = 0$ and $\nu(B') = 0$. Show that the Jordan decomposition is unique if the measures are assumed to be mutually singular.
* If $\mu = \mu_1 - \mu_2$ where μ_1 and μ_2 are positive, show that $\mu_1 \geqslant \mu^+$ and $\mu_2 \geqslant \mu^-$.

Assignment 9: Assigned Wed 10/31. Due Wed 11/07

1. (a) Let ν be a finite (positive) measure. Prove $\nu \ll \mu \iff \forall \varepsilon > 0, \exists \delta > 0 \ni \mu(A) < \delta \implies \nu(A) < \varepsilon$. [This sort of justifies the name "absolutely continuous".]
   (b) Is the previous part true if ν is not finite? Prove or find a counter example.
2. (a) Let ν_1 and ν_2 be two finite signed measures on X. Show that there exists a finite signed measure $\nu_1 \vee \nu_2$ such that $\nu_1 \vee \nu_2(A) \geqslant \nu_1(A) \vee \nu_2(A)$, and for any other finite signed measure ν such that $\nu(A) \geqslant \nu_1(A) \vee \nu_2(A)$ we ust have $\nu_1 \vee \nu_2 \leqslant \nu$.
   (b) If ν_1, ν_2 above are absolutely continuous with respect to a positive σ-finite measure μ, prove $\nu_1 \vee \nu_2 \ll \mu$ and express $\frac{d(\nu_1 \vee \nu_2)}{d\mu}$ in terms of $\frac{d\nu_1}{d\mu}$ and $\frac{d\nu_2}{d\mu}$.
3. Let $(\Omega, \mathcal{F}, P)$ be a measure space with $P(\Omega) = 1$, and $X \in L^1(\Omega, \mathcal{F}, P)$. [The probabilistic interpretation is that Ω is the sample space, $A \in \mathcal{F}$ is an event, X is a random variable, and $P(X \in B)$ is the chance that $X \in B$, where $B \in \mathcal{B}(\mathbb{R})$.]
   (a) Suppose $\mathcal{G} \subseteq \mathcal{F}$ is a σ-sub-algebra of F. Show that there exists a unique $\mathcal{G}$-measurable function Y such that $\int_A Y\,dP = \int_A X\,dP$ for all $A \in \mathcal{G}$. [Y is called the *conditional expection* of X given $\mathcal{G}$, and denoted by $E(X \mid \mathcal{G})$.]
   (b) *(Tower property)* If $\mathcal{H} \subseteq \mathcal{G}$ is a σ-sub-algebra, show that $E(X \mid \mathcal{H}) = E\big(E(X \mid \mathcal{G}) \mid \mathcal{H}\big)$ almost everywhere.
   (c) *(Conditional Jensen)* If $\varphi : \mathbb{R} \to \mathbb{R}$ is convex, show that $\varphi(E(X \mid \mathcal{G})) \leqslant E(\varphi(X) \mid G)$ almost everywhere.
   (d) Suppose $X \in L^2(\Omega, \mathcal{F}, P)$. Show that $E(X \mid \mathcal{G})$ is the L^2-orthogonal projection of X onto the subspace $L^2(\Omega, \mathcal{G})$. [Namely show $E(X \mid \mathcal{G}) \in L^2(\Omega, \mathcal{G})$, and $\int_\Omega (X - E(X \mid G))Y\,dP = 0$ for all $Y \in L^2(\Omega, \mathcal{G})$.]
4. Let μ be a positive measure and ν a finite signed measure. Let $\nu = \nu_{\text{ac}} + \nu_{\text{s}}$ be the Lebesgue decomposition of ν. Show that $\|\nu\| = \|\nu_{\text{ac}}\| + \|\nu_{\text{s}}\|$.
5. Let μ be σ-finite, and define $\varphi : L^\infty \to (L^1)^*$ by $\varphi_g(f) = \int_X fg\,d\mu$. Show that φ is a bijective linear isometry. [In this sense we say L^∞ is the dual of L^1. The reverse identification is not true in general: L^1 can be identified with an *subspace* of $(L^\infty)^*$, but need not be all of it. The proof of this requires the *Hanh-Banach* theorem.]

Optional problems, and details in class I left for you to check.

∗ Show that the Radon Nicodym theorem is not true if ν is σ-finite, but μ is not. Where does the proof we had in class break down if μ is not σ-finite?

∗ Finish the proof of the Lebesgue decomposition (existence and uniqueness) when ν is σ-finite.

∗ If X, Y are Banach spaces show that $B(X, Y)$ with operator norm is a Banach space.

∗ Let $p \in (1, \infty]$, $1/p + 1/q = 1$, and $c < \infty$. If g is a measurable function such that $\sup\{\int_X sg \mid s \text{ is simple, and } \|s\|_p \leqslant 1\} \leqslant c$, show that $g \in L^q$ and $\|g\|_q \leqslant c$.

∗ If μ is a finite signed measure, show that $|\int f\,d\mu| \leqslant \int |f| d|\mu|$.

Assignment 10: Assigned Wed 11/07. Due Wed 11/14

1. (a) Suppose $\sum_{m=1}^\infty (\sum_{n=1}^\infty |a_{m,n}|) < \infty$. Show that $\sum_{m=1}^\infty (\sum_{n=1}^\infty a_{m,n}) = \sum_{n=1}^\infty (\sum_{m=1}^\infty a_{m,n})$.
   (b) Give a counter example to (a) if we only assume $\sum_m \sum_n a_{m,n} < \infty$. Find a counter example where both iterated sums are finite.
2. (a) If X and Y are not σ-finite, show that Fubini's theorem need not hold.
   (b) If $\int_{X \times Y} f\,d(\mu \times \nu)$ is not assumed to exist (in the extended sense), show that both iterated integrals can exist, be finite, but need not be equal.
3. *(Fubini for completions.)* Suppose (X, Σ, μ) and (Y, τ, ν) are two σ-finite, complete measure spaces. Let $\pi = (\Sigma \otimes \tau)_{\mu \times \nu}$ denote the completion of $\Sigma \otimes \tau$ with respect to $\mu \times \nu$.
   (a) Show that $\Sigma \otimes \tau$ need not be $\mu \times \nu$-complete (i.e. $\pi \supsetneq \Sigma \otimes \tau$ in general).
   (b) Suppose $f : X \times Y \to [-\infty, \infty]$ is $\mathcal{F}$-measurable. Define as usual the slices $\varphi_{f,x} : Y \to [0, \infty]$ by $\varphi_{f,x}(y) = f(x, y)$, and similarly $\psi_{f,y}(x) = f(x, y)$. Show that for μ-almost all $x \in X$, $\varphi_{f,x}$ is an τ-measurable, and for ν-almost all y, $\psi_{f,y}$ is an Σ-measurable.
   (c) Suppose f is integrable on $X \times Y$ in the extended sense. Define $F(x) = \int_Y f(x, y)\,d\nu(y)$ and $G(y) = \int_X f(x, y)\,d\mu(x)$. Show F is defined μ-a.e. and Σ-measurable. Similarly show G is defined ν-a.e., and τ-measurable. Further, show and that $\int_X F\,d\mu = \int_Y G\,d\nu = \int_{X \times Y} f d(\mu \times \nu)$.
4. Let (X, Σ, μ), (Y, τ, ν) be two σ-finite measure spaces, $p \in [1, \infty]$, and $f : X \times Y \to \mathbb{R}$ is $\Sigma \otimes \tau$ measurable. Let $F(x) = \int_Y f(x, y)\,d\nu(y)$, and $\psi_{y,f}$ be the slice of f defined by $\psi_{y,f}(x) = f(x, y)$. Show that $\|F\|_{L^p(X)} \leqslant \int_Y \|\psi_{y,f}\|_{L^p(X)}\,d\nu(y)$. [You should verify that when $Y = \{1, 2\}$ with the counting measure, the above is exactly Minkowski's triangle inequality.]
5. For $p \in [1, \infty)$ define $\|f\|_{L^{p,\infty}} = \sup\{\lambda \mu\{|f| > \lambda\}^{1/p} \mid \lambda > 0\}\}$, and the weak L^p space (denoted by $L^{p,\infty}$) by $L^{p,\infty} = \{f \mid \|f\|_{L^{p,\infty}} < \infty\}$. [As usual, we use the convention that functions that are equal almost everywhere are identified with each other.]
   (a) If $f \in L^p$, show $f \in L^{p,\infty}$ and $\|f\|_{L^{p,\infty}} \leqslant \|f\|_p$. Is the converse true?
   (b) If $f, g \in L^{p,\infty}$, show that $f + g \in L^{p,\infty}$. Show further that $\|f + g\|_{L^{p,\infty}} \leqslant c(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}})$ for some constant c independent of f, g. [Thus $\|\cdot\|_{L^{p,\infty}}$ is called a quasi-norm, and $L^{p,\infty}$ is called a quasi-Banach space.]
   (c) If μ is σ-finite, $1 \leqslant p < q < r < \infty$ and $f \in L^{p,\infty} \cap L^{r,\infty}$ then show $f \in L^q$.

Optional problems, and details in class I left for you to check.

∗ Show that the Lebesgue measure on $\mathbb{R}^{m+n}$ is the product of the Lebesgue measurs on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. [Note, you've previously seen that $\mathcal{L}(\mathbb{R}^{m+n}) \supsetneq \mathcal{L}(\mathbb{R}^m) \otimes \mathcal{L}(\mathbb{R}^n)$; however $\mathcal{B}(\mathbb{R}^{m+n}) = \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^n)$.]

∗ For $E \in \Sigma \otimes \tau$, define $f_E(x) = \nu(S_x(E))$ and $g_E(y) = \mu(T_y(E))$. Show that $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ are measurable. [Hint: First assume μ, ν are finite. Let $\Lambda = \{E \mid f_E, g_E \text{ are measurable}\}$. Show that Λ is a λ-system, and Λ contains all rectangles.]

∗ Verify that $\mu \times \nu \overset{\text{def}}{=} \int_X \nu(S_x(E))\,d\mu(x)$ is a measure.

Assignment 11: Assigned Wed 11/14. Due Wed 11/21

1. If $\frac{1}{p}+\frac{1}{q}=1$, $f \in L^p$, $g \in L^q$ show that $f * g$ is bounded and continuous. If $p, q < \infty$, show further $f * g(x) \to 0$ as $|x| \to \infty$.
2. Define $\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \mid \forall m, \alpha, \ \sup_x (1+|x|^m)|D^\alpha f(x)| < \infty\}$. Here $m \in \mathbb{N} \cup \{0\}$, and $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$ is a multi-index, and $D^\alpha f = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f$. The space $\mathcal{S}$ is called the *Schwartz Space.*
   (a) If $p \in [1, \infty)$, $f \in L^p(\mathbb{R}^d)$, $g \in \mathcal{S}(\mathbb{R}^d)$, show that $f * g \in C^\infty(\mathbb{R}^d)$, and further $D^\alpha(f * g) = f * (D^\alpha g)$.
   (b) Show that $\mathcal{S}$ is dense subset of L^p for $p \in [1, \infty)$.
   (c) Show that C_c^∞ is a dense subset of L^p for $p \in [1, \infty)$.
3. (a) If $f, g \in L^2_{\text{per}}([0,1])$, show that $(f * g)^\wedge(n) = \hat{f}(n)\hat{g}(n)$. [Here $L^2_{\text{per}}([0,1])$ denotes (equivalence classes of) all Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{C}$ which have period 1 (i.e. $\tau_1 f = f$), and $\int_0^1 |f|^2 \, d\lambda < \infty$.]
   (b) If $f, g \in L^2([0,1])$, show that $(fg)^\wedge(n) = \hat{f} * \hat{g}(n) \stackrel{\text{def}}{=} \sum_{m \in \mathbb{Z}} \hat{f}(m)\hat{g}(n-m)$.
4. Though I encourage you to check the properties on the Dirichlet and Fejér kernels stated in the optional problems, you may assume them here without proof.
   (a) If $f \in C_{\text{per}}[0,1]$, show that $(\sigma_N f) \to f$ uniformly. [Here $C_{\text{per}}[0,1] = \{f \in C(\mathbb{R}) \mid \tau_1 f = f\}$ denotes all continuous functions with period 1.]

   If $f \in C_{\text{per}}[0,1]$, it turns out that the partial sums $S_N f$ need not converge to f even pointwise. (In fact, there exist many $f \in C_{\text{per}}([0,1])$ such that $S_N f$ is divergent on a dense G_δ in $[0,1]$.) If, however, f is a little bit better than continuous, then the Fourier series of f converges to f pointwise.

   (b) Let $f \in C_{\text{per}}([0,1])$ and $\alpha > 0$. If $\sup_{x,y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty$, then show that $(S_N f) \to f$ pointwise, as $N \to \infty$. [In fact, $(S_N f) \to f$ uniformly.]
5. Let μ be a finite signed Borel measure on $[0,1]$. If $\forall n \in \mathbb{Z}$ $\hat{\mu}(n) = 0$, show $\mu = 0$.

Optional problems, and details in class I left for you to check.

* If $f \in L^p$, $g \in L^q$ with $p, q \in [1, \infty]$ and $1/p + 1/q \geqslant 1$, show that $f * g = g * f$.
* If $f \in L^p$, $g \in L^q$, $h \in L^r$ with $p, q, r \in [1, \infty]$ and $1/p + 1/q + 1/r \geqslant 2$, show that $(f * g) * h = f * (g * h)$.
* Define the Derichlet kernel by $D_N(x) = \sum_{-N}^N \exp(2\pi i n x)$.
  (a) Show that $S_N f(x) = D_N * f(x) \stackrel{\text{def}}{=} \int_0^1 f(y) D_N(x-y) \, dy$. [Recall, $S_N f = \sum_{-N}^N \hat{f}(n) e_n$, where $e_n(x) = e^{2\pi i n x}$, and $\hat{f}(n) = \langle f, e_n \rangle = \int_0^1 f(y) \bar{e}_n(y) \, dy$.]
  (b) Show that $D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}$. Further show $\lim_{N \to \infty} \int_\varepsilon^{1-\varepsilon} |D_N| = \infty$.
* Define Fejér kernel by $F_N = \frac{1}{N} \sum_0^{N-1} D_n$.
  (a) Show that $\sigma_N f \stackrel{\text{def}}{=} \frac{1}{N} \sum_0^{N-1} S_n f = F_N * f$.
  (b) Show that $F_N(x) = \frac{\sin^2(N\pi x)}{N \sin^2(\pi x)}$, and that $\{F_N\}$ is an approximate identity.

Assignment 12: Assigned Wed 11/21. Due Wed 11/28

1. (a) Let $n \in \mathbb{N}$ be even, $\frac{1}{n} + \frac{1}{n'} = 1$. If $\hat{f} \in \ell^{n'}(\mathbb{Z})$, show that $f \in L^n_{\text{per}}([0,1])$ and $\|f\|_{L^n} \leqslant \|\hat{f}\|_{\ell^{n'}}$. [HINT: Let $n = 2m$. Then $\|f\|_{L^n}^n = \|(f^m)^\wedge\|_{\ell^2}^2$.]
   (b) Let $s > \frac{1}{2} - \frac{1}{p} \geqslant 0$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in H^s_{\text{per}}$ show $\hat{f} \in \ell^q(\mathbb{Z})$. Further show that the map $f \mapsto \hat{f}$ is continuous from $H^s_{\text{per}} \to \ell^q$.
   (c) If $n \in \mathbb{N}$ is even, $s > \frac{1}{2} - \frac{1}{n}$ then show that $H^s_{\text{per}} \subseteq L^n([0,1])$ and that the inclusion map is continuous. [This is one part of the Sobolev embedding theorem.]
2. Let $f \in L^2([0,1])$. Show that there exists $u \in C^\infty([0,1] \times (0, \infty))$ such that $u(0,t) = u(1,t)$, $\lim_{t \to 0^+} \|u(\cdot, t) - f(\cdot)\|_2 = 0$, and $\partial_t u - \partial_x^2 u = 0$. [HINT: You may assume the result of the optional problems.]
3. Finish the change of variable proof using the following approach. Recall $U, V \subseteq \mathbb{R}^d$ are open connected sets, and $\varphi : U \to V$ is a C^1 bijection whose inverse is also C^1. Our aim is to show $\lambda(\varphi(A)) = \int_A |\det \nabla\varphi| \, d\lambda$ for all $A \subseteq U$ Borel.

   Assume first that φ, φ^{-1} are both *uniformly* C^1, and U, V are bounded. In this case we showed in class that $\lambda(\varphi(A)) \leqslant \int_A |\det \nabla\varphi| \, d\lambda$ for all Borel $A \subseteq U$.

   (a) If $f : V \to [0, \infty]$ is Borel, show that $\int_V f \leqslant \int_U f \circ \varphi \, |\det \nabla\varphi| \, d\lambda$.
   (b) Show that $\lambda(A) = \int_A |\det \nabla\varphi| \, d\lambda$. [HINT: This follows very quickly previous part.]
   (c) Prove the previous subpart *without* the additional assumptions that φ, φ^{-1} are *uniformly* C^1, and U, V are bounded.

Optional problems, and details in class I left for you to check.

* (a) If $f \in L^1_{\text{per}}([0,1])$, show that $2|\hat{f}(n)| \leqslant \int_0^1 |f(y) - f(y - \frac{1}{2n})| \, dy$.
  (b) Use the previous subpart to give an alternate (perhaps more illuminating) proof of the Riemann Lebesgue lemma.
  (c) If $\alpha \in (0,1)$, $f \in C^\alpha_{\text{per}}([0,1])$, show that $\sup_n |n|^\alpha |f(n)| < \infty$.
  (d) Show by example that the converse of the previous part is false.
* For any $s \geqslant 0$ show that H^s_{per} is a closed subspace of L^2.
* Let $0 \leqslant r \leqslant s$. Show that any bounded sequence in H^s_{per} has a subsequence that is convergent subsequence in H^r_{per}.
* Let $n \in \mathbb{N} \cup \{0\}$, $\alpha \in [0,1)$ $s > 1/2 + n + \alpha$. Show that $H^s_{\text{per}} \subseteq C^{n,\alpha}_{\text{per}}[0,1]$ and the inclusion map is continuous. [Recall $C^{n,\alpha}_{\text{per}}[0,1]$ is the set of all C^n periodic functions on $\mathbb{R}$ (i.e. $\tau_1 f = f$) whose n^{th} derivative is Hölder continuous with exponent α.]
* If $\|\nabla\varphi - I\|_{L^\infty} < \varepsilon$, and $\varphi(0) = 0$, then show that $\varphi((-1,1)^d) \subseteq (-1-d\varepsilon, 1+d\varepsilon)^d$.
* *(Polar Coordinates.)* Let $f \in L^1(\mathbb{R}^2)$. Show that

$$\int_{\mathbb{R}^2} f(x,y) \, dx \, dy = \int_{[0,\infty) \times [0,2\pi)} f(r\cos\theta, r\sin\theta) \, r \, dr \, d\theta$$

Assignment 13: Assigned Wed 11/28. Due Wed 12/05

1. (a) If $f \in L^1(\mathbb{R}^d)$ and f is not identically 0 (a.e.), then show that $Mf \notin L^1(\mathbb{R}^d)$.

   The next few subparts outline a proof that for any $p > 1$, the maximal function is an L^p bounded sublinear operator. Let $p \in (1, \infty)$, $f \in L^p(\mathbb{R}^d)$ and $f \geqslant 0$.

   (b) Show that $\lambda\{Mf > \alpha\} \leqslant \frac{3^d}{(1-\delta)\alpha} \int_{\{f > \delta\alpha\}} f$, for any $t > 0$, $\delta \in (0, 1)$ and $f \geqslant 0$ measurable.

   (c) Let $p \in (1, \infty]$, and $d \in \mathbb{N}$. Show that there exists a constant $c = c(p, d)$ such that $\|Mf\|_p \leqslant c\|f\|_p$ for all $f \in L^p(\mathbb{R}^d)$. [Hint: For $p < \infty$, use the previous part, the identity $\|Mf\|_p^p = \int_0^\infty p\alpha^{p-1}\lambda\{Mf > \alpha\}\, d\alpha$ and optimise in δ.]

2. Let μ be a finite signed Borel measure on $\mathbb{R}^d$. Define $D\mu(x) = \lim_{r \to 0^+} \frac{\mu(B(x,r))}{\lambda(B(x,r))}$.

   (a) If $\mu \perp \lambda$, show that $D\mu = 0$ almost everywhere with respect to λ. [Hint: Write $\mu = \mu_1 + \mu_2$ where $\operatorname{supp}(\mu_1)$ is compact with Lebesgue measure 0, and $\|\mu_2\| < \varepsilon$.]

   (b) If $\mu \perp \lambda$, show that $D|\mu| = \infty$ almost everywhere with respect to μ!

   (c) Show that $D\mu = \frac{d\mu_{ac}}{d\lambda}$ almost everywhere with respect to λ. [Here $\mu = \mu_s + \mu_{ac}$ is the Lebesgue decomposition of μ with respect to λ.]

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a monotone function. Show that f is differentiable almost everywhere. [Hint: Suppose first f is monotone, injective and bounded. Show that $\mu(A) = \lambda(f(A))$ defines a finite Borel measure. How does this help?]

4. If $f, g : [0, 1] \to \mathbb{R}$ are absolutely continuous, then show that fg is absolutely continuous. Conclude $[fg]_0^1 = \int_0^1 f'g + \int_0^1 fg'$.

5. Show that $f : \mathbb{R} \to \mathbb{R}$ is Lipshitz if and only if f is absolutely continuous and $f' \in L^\infty(\mathbb{R})$.

Optional problems, and details in class I left for you to check.

∗ Show that the arbitary union of closed (non-degenerate) cells is Lebesgue measurable.

∗ Find an example of $E \in \mathcal{L}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ such that $\lim_{r \to 0} \frac{\lambda(E \cap B(x,r))}{\lambda(B(x,r))}$ does not exist.

∗ Suppose $f : \mathbb{R} \to \mathbb{R}$ is measurable. Let $\alpha, \beta > 0$ with $\alpha/\beta \notin \mathbb{Q}$. If f has period α, and also has period β (i.e. for all $x \in \mathbb{R}$, $f(x) = f(x + \alpha) = f(x + \beta)$), then show that f is constant almost everywhere. (But f need not be constant everywhere!)

∗ We say the family $\{E_r\}$ *shrinks nicely* to $x \in \mathbb{R}^d$ if there exists $\delta > 0$ such that for all r, $E_r \subseteq B(x, r)$ and $\lambda(E_r) > \delta\lambda(B(x, r))$. If $\{E_r\}$ shrinks nicely to x, show that $\lim \frac{1}{\lambda(E_r)} \int_{E_r} f = f(x)$ for all Lebesgue points of f.

∗ If $f \in L^1(\mathbb{R}^d)$, show that $Mf(x) \geqslant |f(x)|$ at all Lebesgue points of f.

∗ If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then show that f is of bounded variation, and that the variation is absolutely continuous. Conclude f can be written as the difference of two monotone absolutely continuous functions.