Assignment 14: Assigned Wed 12/05. Due never

1. (a) If $\int_{\mathbb{R}^d}(1+|x|)|f(x)|\,dx < \infty$, show that $\hat{f}$ is differentiable and $\partial_j \hat{f}(\xi) = -2\pi i (x_j f(x))^\wedge(\xi)$. [NOTE: $(x_j f(x))^\wedge(\xi)$ means $\hat{g}(\xi)$, where $g(x) = x_j f(x)$.]
   (b) If $f \in C_0^1(\mathbb{R}^d)$, and $\nabla f \in L^1$ show that $(\partial_j f)^\wedge(\xi) = +2\pi i \xi_j \hat{f}(\xi)$.
   (c) Show that the mapping $f \mapsto \hat{f}$ is a bijection in the Schwartz space.
2. If μ is a finite Borel measure on $\mathbb{R}^d$ define $\hat{\mu}(\xi) = \int e^{-2\pi i \langle x, \xi\rangle}\,d\mu(x)$. If $\hat{\mu}(\xi) = 0$ for all ξ, show that $\mu = 0$. [HINT: Show that $\int f\,d\mu = 0$ for all $f \in \mathcal{S}$.]
3. For $f \in L^1$, the formula $\hat{f}(\xi) = \int f(x) e^{-2\pi i\langle x,\xi\rangle}$ allows us to prove many identities: E.g. $(\delta_\lambda f)^\wedge(\xi) = \hat{f}(\lambda\xi)$, etc. For $f \in L^2$, the formula $\hat{f}(\xi) = \int f(x)e^{-2\pi i\langle x,\xi\rangle}$ is no longer valid, as the integral is not defined (in the Lebesgue sense). However, most identites remain valid, and can be proved using an approximation argument. I list a couple here.
   (a) For $f \in L^1$ we know $(\tau_x f)^\wedge(\xi) = e^{-2\pi i\langle x,\xi\rangle}\hat{f}(\xi)$. Prove it for $f \in L^2$.
   (b) Similarly, show that $(\delta_\lambda f)^\wedge(\xi) = \hat{f}(\lambda\xi)$ for all $f \in L^2$.
   (c) Let F denote the Fourier transform operator (i.e. $Ff = \hat{f}$), and R denote the reflection operator (i.e. $Rf(x) = f(-x)$). Note that our Fourier inversion formula (for $f \in L^1$, $\hat{f} \in L^1$) is exactly equivalent to saying $F^2 f = Rf$. Prove $F^2 f = Rf$ for all $f \in L^2$.
4. *(Uncertainty principle)* Suppose $f \in \mathcal{S}(\mathbb{R})$. Show that

$$\Big(\int_{\mathbb{R}} |xf(x)|^2\,dx\Big)\Big(\int_{\mathbb{R}} |\xi\hat{f}(\xi)|^2\,d\xi\Big) \geqslant \frac{1}{16\pi^2}\|f\|_{L^2}^2\|\hat{f}\|_{L^2}^2$$

   [This illustrates a nice localisation principle about the Fourier transform. The first integral measures the spread of the function f. The second the spread of the Fourier transform $\hat{f}$. Here you show that this product is bounded below! Thus, if one is locallized the other is forced to be spread out.

   In quantum mechanics, Haysenberg's uncertainty principle says the product of errors in measuring the position and momentum (respectively) of a particle is bounded below. The proof, once you know enough Physics, reduces to the above inequality.

   Hint: Consider $\int_{\mathbb{R}} xf(x)f'(x)\,dx$.]
5. *(Central limit theorem)* Let $f \in L^1(\mathbb{R})$ be such that $f \geqslant 0$ and $\int x^2 f(x)\,dx < \infty$. Define $g_n = (f * \cdots * f)$ (n-times), and $h_n(x) = \delta_{1/\sqrt{n}} g_n(x) = \sqrt{n} g_n(\sqrt{n}x)$. Show

$$\hat{h}_n(\xi) \xrightarrow{n\to\infty} \exp\big(-2\pi i\mu\xi - 2\pi^2 i\sigma^2\xi^2\big),$$

   where $\mu = \int xf(x)\,dx$ and $\sigma^2 = \int (x-\mu)^2 f(x)\,dx$. [The central limit theorem says that tabulating results of a large number of independent trials of any experiment produces a "bell curve". The key step in the proof, which you will no doubt see next semester, is showing that any function convoved with itself often enough looks like a Gaussian.]
6. *(Sobolev spaces)* For $f \in L^2(\mathbb{R}^d)$ and $s \geqslant 0$ define

$$\|f\|_{H^s}^2 = \int (1+|\xi|^s)^2 |\hat{f}(\xi)|^2\,d\xi, \qquad \text{and} \qquad H^s = \{f \in L^2 \mid \|f\|_{H^s} < \infty\}.$$

   Intuitively, we think of H^s as the space of functions with "s" "weak-derivatives" in L^2. (This will be formalized in your functional analysis course.)
   (a) If $f \in C_0^n(\mathbb{R}^d)$ and $D^\alpha f \in L^2$ for all $|\alpha| < n$, then show that $f \in H^n(\mathbb{R}^d)$.
   (b) For $s \in (0,1]$ show that there exists a constant c such that for all $x \in \mathbb{R}^d$, and $f \in H^s$ we have $\|f - \tau_x f\|_{L^2} \leqslant c|x|^s \|f\|_{H^s}$.
7. *(Sobolev embedding)* If $n \in \mathbb{N}$ and $f \in H^s(\mathbb{R}^d)$ for $s > n + \frac{d}{2}$ then show that $f \in C^n$, and further the inclusion map $H^s \to C^n$ is continuous.
8. (a) *(Elliptic regularity)* Let $Lu = \sum a_{ij}\partial_i\partial_j u + \sum b_i \partial_i u + cu$, where a_{ij}, b_i, c are constants. Suppose $\exists \lambda > 0$ such that $a_{ij} = a_{ji}$ and $|\sum a_{ij}\xi_i\xi_j| \geqslant \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^n$ (this assumption is called ellipticity). If $f \in C^\infty$, $D^\alpha f \in L^1$ for all α, and $u, \partial_i u, \partial_i\partial_j u \in L^1 \cap C^0$ are such that $Lu = f$, show that $u \in C^\infty$. [To emphasize why this is surprising, choose for example $L = \triangle$. Then $\triangle u = f$ makes no mention of a mixed derivative $\partial_1\partial_2 u$. Yet, all such mixed derivatives exist and are smooth. Hint: If $f \in H^s$ show that $u \in H^{s+2}$.]
   (b) Show by example that the previous subpart is false without the ellipticity assumption.
9. *(Trace theorems)* Let $p \in \mathbb{R}^m$ be fixed. Given $f : \mathbb{R}^{m+n} \to \mathbb{R}$ define $S_p f : \mathbb{R}^n \to \mathbb{R}$ by $S_p f(y) = f(p, y)$.
   (a) Let $s > m/2$, and $s' = s - m/2$. Show that there exists a constant c such that $\|S_p f\|_{H^{s'}(\mathbb{R}^n)} \leqslant c\|f\|_{H^s(\mathbb{R}^{m+n})}$.
   (b) Show that the section operator S_p extends to a continuous linear operator from $H^s(\mathbb{R}^{m+n})$ to $H^{s'}(\mathbb{R}^n)$. [Given an arbitrary L^2 function on $\mathbb{R}^{m+n}$ it is of course impossible to restrict it to an m-dimensional hyper-plane. However, if your function has more than $n/2$ "Sobolev derivatives", then you can make sense of this restriction, and the restriction still has $s - n/2$ "Sobolev derivatives".]
10. *(Reliech Lemma)* Let $K \subseteq \mathbb{R}^d$ be compact, $0 \leqslant s_1 < s_2$, and suppose $\{f_n\}$ are a sequence of functions supported in K. If the sequence $\{f_n\}$ is bounded in H^{s_2}, then show that it has a convergent subsequence in H^{s_1}. [This is the generalization of the Arzella-Ascolli theorem in this context.]