21-372 PDE: Midterm 1.

Feb 20^{th}, 2012

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 50 mins. The exam has a total of 4 questions and 20 points.*
- *You may use any result from class or homework* ***PROVIDED*** *it was proved* independently *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*
- *The questions are all roughly the same length and difficulty; however depending on your intuition you might find some easier than others.*

[5] 1. Find the general solution of the PDE $y\partial_x u - x\partial_y u = x^2y^3 + y^5$

[5] 2. Let D be a bounded region in $\mathbb{R}^3$, and u satisfy the PDE $\partial_t^2 u - \triangle u = \partial_t u$ with Dirichlet boundary conditions $u = 0$ on ∂D. Define E to be the volume integral

$$E(t) = \int_D (\partial_t u)^2 + |\nabla u|^2 \, dV.$$

What is the sign of $\frac{dE}{dt}$? Prove it.

[2] 3. (a) Find all functions f such that the function $u(x,t) = f(t)\sin(x)$ satisfies the PDE

$$\partial_t u - \partial_x^2 u = 0. \tag{1}$$

[3] (b) Suppose u satisfies the PDE (1) above, for $x \in (0,\pi)$, $t > 0$ with boundary conditions $u(0,t) = u(\pi,t) = 0$ and initial data $u(x,0) = \frac{1}{372}$. Note u does not satisfy the boundary conditions at time $t = 0$; but it will satisfy the boundary conditions for all $t > 0$. Show that for all $t > 0$ and $x \in (0,\pi)$ we must have $u(x,t) > 0$. [You may *NOT* use the strong maximum/minimum principle for this question, as we have not yet proved it in this context. However, you may freely use the weak maximum principle and the previous subpart (hint hint).]

[5] 4. Suppose u satisfies the PDE $\partial_t u - \frac{1}{2}\partial_x^2 u = 0$ for $x \in \mathbb{R}$ and $t > 0$, on $\mathbb{R}$, with initial data $u(x,0) = f(x)$. Suppose further f is such that $\int_{-\infty}^{\infty} |f| = 1$. For any $x \in \mathbb{R}$, must $\lim_{t\to\infty} u(x,t)$ exist? If yes, what is it's value? Prove your answer. [NOTE: If you are concerned about boundary conditions, you may assume $\lim_{x\to\pm\infty} u(x,t) = 0$.]