Math 372: PDE Homework.

The problem numbers refer to problems from your text book (second edition). I will often assign problems which are not in the text book. *Keep in mind that there is a firm 'no late homework' policy.*

Assignment 1: Assigned Wed 01/16. Due Wed 01/23

1. **Sec. 1.1.** 3, 4
2. **Sec. 1.2.** 2, 3, 11, 12 [You don't need the 'coordinate method' to do 11, as the hint suggests. It can be done directly using the method of characteristics.]
3. Find the general solutions of the PDEs
   (a) $(1 + x^2)\partial_x u + \partial_y u = yu^2$.
   (b) $(1-xy)\partial_x u+\partial_y u+\partial_z u = (y-z)u$.
4. *(Optional, harder)* Given that temperature in a conductor obeys the equation $\partial_t u - \triangle u = 0$, formulate a rigorous version of the statement *"heat does not collect at hot points."* Prove it!

Assignment 2: Assigned Wed 01/23. Due Wed 01/30

1. **Sec. 1.3.** 1, 2, 10, 11
2. A fluid is called *incompressible* if for any region R the rate at which the fluid enters and leaves the region is 0. If an incompressible the fluid has a constant density, show that its velocity field must be divergence free. [Once you figure out how to translate the above into mathematical symbols, the solution will be evident.]
3. Newtons law of cooling says that a body loses heat to it's surroundings at a rate proportional to the temperature difference. Consider a thin (1D) wire immersed in a medium of constant temperature θ_0, which exchanges heat with the surroundings according to Newtons law. Find a PDE satisfied by the temperature in the wire.
4. Let $D \subseteq \mathbb{R}^3$ be the region occupied by a fluid body (e.g. a lake), $u(x, t)$ be the instantaneous velocity of the fluid at point $x \in D$ and time t, and $\rho(x, t)$ be the concentration of some pollutant at time t and position $x \in \mathbb{R}^3$. Fick's law says that the rate of flow of the pollutant is proportional to the concentration gradient. Use this to derive a PDE for ρ. [This is called the advection diffusion equation.]
5. For this problem, identify $(x, y) \in \mathbb{R}^2$ with the complex number $x + iy \in \mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *holomorphic* if for every $z \in \mathbb{C}$, $\lim\limits_{\zeta\to 0} \frac{f(z+\zeta)-f(z)}{\zeta}$ exists. Note: The limit is taken as $\zeta \in \mathbb{C}$ approaches 0.
   (a) If f is holomorphic, show that $\partial_x f = -i\partial_y f$. Conclude $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$, where $u = \operatorname{Re} f$ and $v = \operatorname{im} f$.
   (b) If f is holomorphic, let $f' = \lim\limits_{\zeta\to 0} \frac{f(z+\zeta)-f(z)}{\zeta}$ denote the "complex" derivative of f. Amazingly, if f is holomorphic, then f' is also holomorphic! (In contrast, if f is only differentiable, f' need not even be continuous, let alone differentiable.) Assuming f and f' are holomorphic, show that $u \stackrel{\text{def}}{=} \operatorname{Re} f$ is harmonic (i.e. show $\triangle u = 0$).
6. *(Optional, slightly harder)* Given that the profile of a string obeys the wave equation $\partial_t^2 u - c^2\partial_x^2 u = 0$, make rigorous and prove the statement: *"The speed of propagation is c."*
7. *(Optional, HARD)* This is a follow up to last times optional question. Many of you translated the statement "Heat does not collect at hot points" into the statement that the temperature at a *strict* local maximum must decrease with time. Question: What can you say about the *global* maximum of temperature? Keep in mind that a global maximum need not be a *strict* local maximum.
8. *(Optional, CHALLENGE)* Harmonic functions have many beautiful properties. One surprising one is the mean value property: If $u : \mathbb{R}^d \to \mathbb{R}$ is such that $\triangle u = 0$, then $u(0) = \frac{1}{\operatorname{area}(\partial B_R)} \int_{B_R} u \, dS$, where B_R is the ball with center 0 and radius R. [This holds in all dimensions. It is easy to prove in one dimension. If you know some complex analysis, you can deduce this quickly from the Cauchy integral formula in 2 dimensions. It is a CLEVER application of the divergence theorem in higher dimensions. (I will do this in class when I study harmonic functions.)]

Assignment 3: Assigned Wed 01/30. Due Wed 02/06

1. **Sec. 1.4.** 3, 4, 6
2. **Sec. 2.1.** 5, 10.
3. (a) Suppose u satisfies $u_{tt} - c^2 u_{xx} = 0$ on the interval (a, b), for $t > 0$. Under what boundary conditions on u is energy is conserved? Prove it, and provide some physical explanation.

   (b) Let D be some region in (2 or 3) dimensional space. Suppose u satisfies $u_{tt} - c^2 \triangle u = 0$ in the region D. Define the energy to be the area/volume integral $E = \int_D (u_t^2 + c^2 |\nabla u|^2)\, dv$. Under what boundary conditions on u is energy conserved? Prove it. [Hint: From the previous part you should be able to guess the boundary conditions. For the proof, it requires replacing 'integration by parts' from your previous proof by a clever application of the divergence theorem.]
4. Let D be a region in $\mathbb{R}^3$, and $c, r > 0$ be constants, and a, f be functions depending only on the spatial variables x_1, x_2 and x_3 such that $a(x) \geqslant 0$ for all $x \in D$. Show that solutions to the PDE

$$\partial_t^2 u - c^2 \triangle u + au + r\partial_t u = f$$

   with Dirichlet boundary conditions $u = 0$ on ∂D, and initial data

$$u(x, 0) = \varphi(x), \qquad \partial_t u(x, 0) = \psi(x)$$

   are unique. That is, if u_1 and u_2 are two solutions to the above PDE, with the same boundary conditions and initial data, show that they are equal. [Hint: Suppose u_1 and u_2 are two solutions. Set $v = u_1 - u_2$. Now try and cook up some 'energy' which will help you show v is 0. Note, the energy you cook up (if you do it right) won't be conserved! It will however decrease with time.]
5. Solitary waves (or solitons) are waves that travel great distances without changing shape. Tsunami's are one example. Scientific study began with Scott Russell in 1834, who followed such a wave in a channel on horseback, and was fascinated by it's rapid pace and unchanging shape. In 1895, Kortweg and De Vries showed that the evolution of the profile is governed by the equation

$$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0.$$

   For this question, suppose u is a solution to the above equation for $x \in \mathbb{R}$, $t > 0$. Suppose further that u and all derivatives (including higher order derivatives) of u decay to 0 as $x \to \pm\infty$.

   (a) Let $p = \int_{-\infty}^{\infty} u(x, t)\, dx$. Show that p is constant in time. [Physically, p is the momentum of the wave.]

   (b) Let $E = \int_{-\infty}^{\infty} u(x, t)^2\, dx$. Show that E is constant in time. [Physically, E is the energy of the wave.]

   (c) It turns out that the KdV equation has *infinitely many* conserved quantities. The energy and momentum above are the *only two* which have any physical meaning. Can you find a non-trivial conserved quantity that's not a linear combination of p and E?

Assignment 4: Assigned Wed 02/06. Due Wed 02/13

1. **Sec. 2.3.** 3, 5, 7.
2. Suppose u satisfies the wave equation $\partial_t^2 u - \partial_x^2 u = 0$ for $x \in \mathbb{R}$, $t > 0$. Let $x \in \mathbb{R}$, $t > 0$, and $a, b \in (0, t)$. True or false:

$$u(x - a, t - b) + u(x + a, t + b) = u(x - b, t - a) + u(x + b, t + a)?$$

   If true, prove it. If false, find a counter example.
3. Let $L, T > 0$, and a, b be two continuous functions such that $a(x, t) \geqslant 0$ (with no sign assumption on b).

   (a) If u is continuous on the closed rectangle $[0, L] \times [0, T]$ and satisfies $\partial_t u - a\partial_x^2 u + b\partial_x u < 0$ on the open rectangle $(0, L) \times (0, T)$, then show that u attains it's maximum *only* on the sides or bottom of this rectangle.

   (b) If instead $\partial_t u - a\partial_x^2 u + b\partial_x u \leqslant 0$. Show that u attains it's maximum on the sides or bottom of the above rectangle.

   (c) Show that solutions to the PDE $\partial_t u - a\partial_x^2 u + b\partial_x u = f$, with initial data $u(x, 0) = \varphi(x)$, and Dirichlet boundary conditions $u(0, t) = g_1(t)$ and $u(L, t) = g_2(t)$ are unique. [If a is not constant in x, then you won't (easily) be able to prove uniqueness to this PDE using energy methods.]
4. Let a, b, L, T be as in the previous problem, and u is continuous on the rectangle $\bar{R} = [0, L] \times [0, T]$.

   (a) Suppose c is a function such that $c(x, t) \geqslant 0$ for all $x \in (0, L)$ and $t \in (0, T)$ and $\partial_t u - a\partial_x^2 u + b\partial_x u + cu \leqslant 0$. Show that if u attains a *non-negative* maximum, then it must be on the sides or bottom of the rectangle $\bar{R}$.

   (b) In the previous part, show by example that a *negative* maximum of u can be attained in the interior (or top) of $\bar{R}$.

   (c) Instead of assuming $c(x, t) \geqslant 0$, suppose we assume $c(x, t) \geqslant -M$ for some constant M. Then if $u \leqslant 0$ on the sides and bottom of $\bar{R}$, must $u \leqslant 0$ on all of R? Prove or find a counter example.
5. Let u satisfy the equation $\partial_t u - \partial_x^2 u = u(1 - u)$ on the space-time rectangle $R = (0, L) \times (0, T)$, and is continuous on $\bar{R}$. If $0 \leqslant u \leqslant 1$ on the sides and bottom of R, must $0 \leqslant u \leqslant 1$ on all of R? Prove it. [A standard fact in analysis will guarantee $-M \leqslant u \leqslant M$ for some big number M. Feel free to assume this if you haven't seen it before. Physically, this is the *reaction diffusion equation* used to describe evolution of temperature in an exothermic reaction (e.g. burning fuel).]
6. *(Optional, challenge.)* If $\partial_t u - \kappa\partial_x^2 u \leqslant 0$ in the space time rectangle $R = (0, L) \times (0, T)$. If u attains an interior maximum at a point (x_0, t_0) show that u must be constant up to time t_0. [This is the *Strong Maximum Principle*, and has an elementary (but TRICKY) proof using only the weak maximum principle. I will offer a reward for a correct solution.]

Assignment 5: Assigned Wed 02/13. Due Fri 02/22

1. Solve $u_t - \frac{1}{2}u_{xx} = 0$ on the line, with initial data $u(x,0) = |x|$. Sketch profiles of u for $t = \frac{1}{2}$, $t = 1$, $t = 10$. [This problem will show you how the corners of the initial data get smoothed out.]
2. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, define $G(x,t) = (2\pi t)^{-n/2} e^{-|x|^2/2t}$.
   (a) Show that $\partial_t G = \frac{1}{2}\triangle G$ for any $t > 0$.
   (b) Show that $\int_{\mathbb{R}^n} G(x,t)\,dx = 1$ for any $t > 0$. [By $\int_{\mathbb{R}^n} G(x,t)\,dx$, I mean the iterated integral $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G(x_1, \dots, x_n, t)\,dx_1 \cdots dx_n$.]
   (c) Write down a formula for a solution to the heat equation $\partial_t u - \frac{1}{2}\triangle u = 0$, for $x \in \mathbb{R}^n, t > 0$ with initial data $u(x,0) = f(x)$. Verify your formula solves the equation for $t > 0$. (Verifying that it has the correct initial data is harder, and will be handled later on.)
3. (a) Compute the solution of $\partial_t u - \frac{1}{2}\partial_{xx}^2 u = f$ given $u(x,0) = 0$, and $f(x,t) = 1$ if $|x| \leqslant 1$, and $f(x,t) = 0$ otherwise.
   (b) If $\alpha \geqslant 0$ and $|x| > 1$, compute $\lim_{t\to\infty} \frac{1}{t^\alpha} u(x,t)$.

Assignment 6: Assigned Fri 02/22. Due Wed 02/27

1. Let f be a function, and suppose $u(x,t) = \int_{-\infty}^{\infty} f(y)G(x-y,t)\,dy$.
   (a) If f is bounded and $\lim_{x\to\pm\infty} f(x) = 0$, show that for every $t > 0$ we also have $\lim_{x\to\pm\infty} u(x,t) = 0$.
   (b) If instead $\int_{-\infty}^{\infty} |f| < \infty$, must we still have $\lim_{x\to\pm\infty} u(x,t) = 0$? Prove it.
2. Let f be a bounded function, and $u(x,t) = \int_{-\infty}^{\infty} f(y)G(x-y,t)\,dy$.
   (a) Show that $\lim_{t\to 0^+} u(x,t) = \frac{1}{2}(f(x^+) + f(x^-))$, where $f(x^\pm) = \lim_{y\to x_\pm} f(y)$.
   (b) (Unrelated) If f is differentiable, f' is bounded and f' is continuous at x, show that $\lim_{t\to 0^+} \partial_x u(x,t) = f'(x)$.
3. Suppose we want to solve $\partial_t u - \kappa\partial_x^2 u = 0$ for $x \in (0,\infty)$, $t > 0$ with $u(x,0) = f(x)$ and Dirichlet boundary conditions $u(0,t) = 0$. Here's a trick:
   (a) Let $g(x) = f(x)$ for $x \geqslant 0$, and $g(x) = -f(-x)$ for $x < 0$ (this is called the *odd* extension of f). Let v be the solution of $\partial_t v - \kappa\partial_x^2 v = 0$ for $x \in \mathbb{R}$, $t > 0$ with initial data $v(x,0) = g(x)$. Show that $v(0,t) = 0$ for all $t > 0$. [You may, but need not, use the explicit solution formula. You may assume all functions decay at infinity.]
   (b) Use the explicit formula for v from the previous subpart to show that $u(x,t) = \int_0^\infty f(y)[G(x-y, 2\kappa t) - G(x+y, 2\kappa t)]\,dy$ is the desired solution.
   (c) Find an explicit formula for the solution of the PDE $\partial_t u - \kappa\partial_x^2 u = g(x,t)$, for $x > 0, t > 0$, with Dirichlet boundary conditions $u(0,t) = 0$ and initial data $u(x,0) = f(x)$. [You may assume decay as $x \to +\infty$.]
4. (a) Use the same trick we used for the heat equation to find an explicit formula for the solution of the PDE $\partial_t^2 u - c^2\partial_x^2 u = 0$, for $x > 0, t > 0$, with Dirichlet boundary conditions $u(0,t) = 0$ and initial data $u(x,0) = \varphi(x)$, and $\partial_t u(x,0) = \psi(x)$.
   (b) For the previous subpart, sketch the domain of dependence of a point (x,t). [Do two cases: $x < ct$ and $x \geqslant ct$. Your pictures will be different!]
   (c) Appropriately modify the previous trick to find an explicit formula for the solution of the PDE $\partial_t^2 u - c^2\partial_x^2 u = g(x,t)$, for $x > 0, t > 0$, with Neumann boundary conditions $\partial_x u(0,t) = 0$ and initial data $u(x,0) = \varphi(x)$, and $\partial_t u(x,0) = \psi(x)$.

Assignment 7: Assigned Wed 02/27. Due Wed 03/06

1. **Sec. 4.1.** 3, 4, 6.
2. **Sec. 4.2.** 2, 4.

If you can do both the problems below *PERFECTLY*, then you don't have to turn in the book problems for this assignment.

3. This problem outlines a proof of the strong Minimum principle. Suppose $\partial_t u - \kappa\partial_x^2 u = 0$ in the domain $R = (a,b) \times (0,T)$, and u is continuous up to the boundary of R. Suppose that the minimum of u on the rectangle R is 0 and is attained at some point (x_0, t_0) in the *interior* of R.
   (a) Show that $u(x_0, s) = 0$ for all $s \leqslant t_0$. [HINT: Q#3 on your exam helps (a lot).]
   (b) Show that $u(y, t_0) = 0$ for all $y \in [a,b]$.
   (c) Conclude $u(x,s) = 0$ for all $x \in [a,b]$ and $s \leqslant t_0$.
4. Suppose $\lambda \in \mathbb{R}$ and X is a function such that $X(0) = X(L) = 0$, $X'' = -\lambda X$ and $X(x) > 0$ for all $x \in (0,L)$. Show that for *any* $f \in C^2$ such that $f(0) = f(L) = 0$ we must have
$$\lambda \leqslant \frac{\int_0^L (f')^2}{\int_0^L f^2}.$$

Assignment 8: Assigned Wed 03/06. Due Wed 03/20

1. **Sec. 5.1.** 2, 5
2. This problem extends the symmetry and orthogonality lemmas to higher dimensions. Let D be a bounded region in $\mathbb{R}^3$ (or in $\mathbb{R}^2$).
   (a) Let u, v be two functions. Show that $\int_D u \triangle v = \int_{\partial D} u \frac{\partial v}{\partial \hat{n}} - \int_D (\nabla u) \cdot (\nabla v)$. [Recall, by $\int_D f$, I mean the volume integral $\iiint_D f(x,y,z)\,dV$. Similarly by $\int_{\partial D} f$, I mean the surface integral $\iint_D f(x,y,z)\,dS$.]
   (b) *(Positivity)* Suppose $-\triangle u = \lambda u$ in D, with the Dirichlet boundary condition $u = 0$ on the boundary of D. Show that $\lambda = \left(\int_D |\nabla u|^2\right) / \left(\int_D u^2\right)$, and hence $\lambda \geqslant 0$.
   (c) *(Symmetry)* Suppose u and v satisfy the Dirichlet boundary conditions $u = v = 0$ on the boundary of D. Show that $\int_D (-\triangle u) v = \int_D u(-\triangle v)$.
   (d) *(Orthogonality)* Suppose $-\triangle u = \lambda u$, $-\triangle v = \mu v$, $\lambda \neq \mu$, and u, v satisfy the Dirichlet boundary conditions $u = v = 0$ on the boundary of D. Show that $\int_D uv = 0$.
   (e) Do the previous subparts work if we replace the Dirichlet boundary conditions with Neumann? Explain.
3. If f is a real valued function, and c_n be the n^{th} complex Fourier coefficient of f. Show that $c_{-n} = \overline{c_n}$.
4. A file on the website contains a mix of two audio clips: One of birds chirping, and the other of a flute. Your task is to remove the chirping birds, and recover (as well as you can) the original clip of the flute alone.

   You should email your final code `prakjitj@andrew.cmu.edu`, along with instructions on how to run it; when given the WAV file as input, it should produce a WAV file as output. To measure the error, we will convert your wave file to a vector, and normalise it to have length 1. The original wave file for the flute will also be converted it to a vector (again normalised to have length 1). The error will be half the distance between the above two normalised vectors. [The distance (as measured above) between the mixed clip and the original flute sample is about 0.1362. My code currently reduces this distance to 0.0045, at which point the birds are essentially inaudible, with very little "artifacting". See if you can do better!]

Assignment 9: Assigned Wed 03/20. Due Wed 03/27

1. **Sec. 5.4.** 1, 7, 13
2. Find a sequence of functions (f_n) such that $\int_{-\infty}^{\infty} |f_n(x)|^2\,dx < \infty$, $(f_n) \to 0$ uniformly on $(-\infty, \infty)$, however (f_n) does *not* converge to 0 in $L^2(-\infty, \infty)$. Why does this not contradict the result from class?
3. Suppose f is a piecewise differentiable $2L$-periodic (complex valued) function. Let c_n be the (Complex) Fourier coefficients of f, and d_n be the (complex) Fourier coefficients of f'.
   (a) Assuming that the Fourier series of f and f' converge, and that the Fourier series of f can be differentiated term by term, guess a relation between the c_n's, and d_n's.
   (b) Prove your relation above *without* assuming term by term differentiation of the Fourier series, or convergence of the above Fourier series. [You may assume that f' is continuous.]
   (c) As I mentioned in class, differentiability of a function usually translates to faster decay of it's Fourier coefficients. Here's one example: Usually if one only knew $\int -L^L |f|^2 < \infty$, then we only know (by Bessel's inequality) that $\sum_{-\infty}^{\infty} |c_n|^2 < \infty$. If additionally $\int_{-L}^{L} |f'|^2 < \infty$, show that $\sum_{-\infty}^{\infty} |nc_n|^2 < \infty$
   (d) If $\int_{-L}^{L} |f|^2 < \infty$ and $\int_{-L}^{L} |f'|^2$ show that $|c_n| \leqslant \frac{k}{n}$ for some constant k, independent of n. [Warning: f need not be bounded! You may, however, assume the relation between c_n and d_n you derived in the previous parts holds.]

Assignment 10: Assigned Wed 03/27. Due Wed 04/03

1. Let f be a complex valued $2L$-periodic function, and $c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\frac{n\pi}{L}x}\,dx$ be the n^{th} Complex Fourier coefficient of f. Let $S_N f = \sum_{-N}^{N} c_n f_n$ be the partial sums and $\sigma_N f = \frac{1}{N} \sum_{0}^{N-1} S_N f$.
   (a) Show that $\sigma_N f(x) = \int_{-L}^{L} K_N(x-y) f(y)\,dy$, where $K_N(z) = \frac{\sin\left(\frac{N}{2}\frac{\pi}{L}x\right)^2}{2NL\sin\left(\frac{1}{2}\frac{\pi}{L}x\right)^2}$.
   (b) Show that there exists a constant $C > 0$ such that for all N, x, we have $K_N(x) \leqslant \frac{C}{N} \min\{N^2, \frac{1}{x^2}\}$.
   (c) For all N, show that $K_N \geqslant 0$, and $\int_{-L}^{L} K_N(x)\,dx = 1$.
   (d) For any $\varepsilon > 0$, show that $\lim_{N\to\infty} \left(\int_{-L}^{-\varepsilon} K_N(x)\,dx + \int_{\varepsilon}^{L} K_N(x)\,dx\right) = 0$.
   (e) If f is bounded and continuous at the point $x \in [-L, L]$, then show that $\lim\limits_{N\to\infty} \sigma_N f(x) = f(x)$. [If you know uniform continuity, this proof will also show $\sigma_N f$ will converge to f uniformly.]
2. Let f be a continuous function on $[0, L]$, and A_n be the Fourier Sine coefficients of f. The Sobolev embedding theorem says that if for some $k \in \mathbb{N}$ and $s > \frac{1}{2}$ we have $\sum |n^{k+s} A_n|^2 < \infty$, then f is k-times differentiable, and its k-th derivative is continuous. [As I've said in class, better differentiability of the function translates to faster decay of the Fourier coefficients. This is a theorem illustrating this principle.]
   (a) *(Optional)* Prove the above version of the Sobolev embedding theorem. [Hint: For $k = 1$, use the Cauchy-Schwarz inequality to show that $\sum |nA_n| < \infty$. If you don't know the "usual" theorems guaranteeing differentiability of limits, you'll have to resort to the Weierstrass M-test and the Mean value theorem.]
   (b) ***(Not** optional)* Suppose u satisfies the heat equation on $[0, L]$ with Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$ and initial data $u(x,0) = f(x)$. Show that for any $t > 0$ the function u is infinitely differentiable as a function of x.
   (c) Explain why your proof above won't work for the wave equation.

Assignment 11: Assigned Wed 04/03. Due Wed 04/10

1. (a) Suppose $\int_{-L}^{L}|g| < \infty$. Show $\lim_{N\to\infty}\int_{-L}^{L}\sin((N+\frac{1}{2})\frac{\pi x}{L})g(x)\,dx = 0$.

   Let $x \in (-L, L)$, $\alpha \in (0, 1]$. We say a function f is Hölder continuous at x with exponent α if there exists a constant $C > 0$ such that $|f(y) - f(x)| \leqslant C|y - x|^{\alpha}$. The next two problems show that Hölder continuity implies continuity, but not conversely.

   (b) *(Optional.)* If for some $\alpha \in (0, 1]$ the function f is Hölder continuous at x, then f is continuous at x.

   (c) Find a function f such that f is continuous at 0, however for EVERY $\alpha \in (0, 1]$ the function f is NOT Hölder continuous at 0 with exponent α.

   (d) Let $x \in (-L, L)$ and $\alpha \in (0, 1]$. Suppose f is a function such that $\int_{-L}^{L}|f| < \infty$, and f is Hölder continuous at x with exponent α. Define the function h by

   $$h(y) = \frac{f(y) - f(x)}{\sin(\frac{\pi(x-y)}{2L})}$$

   Show that $\int_{-L}^{L}|h| < \infty$.

   (e) Let x, f, α be as in the previous part. Show that the complex Fourier series of f converges to $f(x)$ at the point x.

   (f) Let f and $x \in (-L, L)$ be such that $\int_{-L}^{L}|f| < \infty$ and both the left and right derivatives of f exist and are finite (but are not necessarily equal) at x. Show that the Fourier series of f converges to $f(x)$ at the point x.

2. Suppose u satisfies the heat equation $\partial_t u - \partial_x^2 u = 0$ for $x \in (0, L)$ and $t > 0$ with Neumann boundary conditions $\partial_x u(0, t) = \partial_x u(L, t) = 0$ and initial data $u(x, 0) = f(x)$. You may assume $\|f\| < \infty$.

   Consider now the limit of the functions $u(\cdot, t)$ as $t \to \infty$. Namely, for any fixed $t > 0$, view the slice of u at time t as a function of x. Then consider the limit of these functions as $t \to \infty$. Does this limit exist in the pointwise, uniform or L^2 sense? Prove it. Also compute the limit.

3. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain. Let f and g be two given functions. Show that the solution to the Poisson equation $-\triangle u = f$ for $x \in \Omega$, with Dirichlet boundary conditions $u = g$ for $x \in \partial\Omega$ is unique.

4. *(Optional)* Verify the following identities in polar coordinates:

   (a) $\partial_x r\,\hat{x} + \partial_y r\,\hat{y} = \hat{r}$

   (b) $\partial_x \theta\,\hat{x} + \partial_y \theta\,\hat{y} = \frac{1}{r}\hat{\theta}$

   (c) $\nabla \cdot \hat{r} = \frac{1}{r}$.

   (d) $\nabla \cdot \hat{\theta} = 0$.

Assignment 12: Assigned Wed 04/10. Due Wed 04/17

1. **Sec. 6.1.** 6, 9.

2. Let $P(r, \theta)$ be the Poisson kernel on a disk of radius a. For any $\varepsilon > 0$, show that $\lim_{r\to a^-}\left[\int_{-\pi}^{-\varepsilon} P(r, \theta)\,d\theta + \int_{\varepsilon}^{\pi} P(r, \theta)\right] = 0$.

3. Let D be a disk of radius a, and u be the solution of $-\triangle u = 0$ in D, with boundary condition $u(a, \theta) = f(\theta)$. Suppose $\int_{-\pi}^{\pi}|f(\theta)|^2\,d\theta < \infty$. Let $c_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(r, \theta)e^{-in\pi\theta}\,d\theta$ be the complex Fourier coefficients of $u(r, \cdot)$.

   (a) For any $s \geqslant 0$ and $r < a$, show that $\sum_{-\infty}^{\infty}|n^s c_n(r)| < \infty$. [As before, you know $c_n(r)$ explicitly in terms of $c_n(a)$.]

   (b) Show that for any $r < a$, u is infinitely differentiable.

   (c) Show that $\lim_{r\to a^-}\int_{-\pi}^{\pi}|u(r, \theta) - f(\theta)|^2\,d\theta = 0$. [This is one situation where the theorem allowing you to interchange the limit and integral *does not* apply. You'll have to do this out explicitly. Hint – Fourier series . . . , but perhaps you guessed that already.]

4. *(Separation of variables on an annulus)* Given $a, b \in \mathbb{R}$ such that $0 < a < b$, let the annulus A be defined by $A = \{x \in \mathbb{R}^2 \mid a < |x| < b\}$. Given two periodic functions f, g, find a function u such that $\triangle u = 0$ in A, with boundary conditions $u(a, \theta) = f(\theta)$ and $u(b, \theta) = g(\theta)$. You may leave your answer as an infinite series involving the appropriate Fourier coefficients of f and g.

5. *(Optional challenge that will earn you a reward.)* Suppose $\triangle u = 0$ in Ω and u attains it's maximum at some point $y \in \partial\Omega$. If u is non-constant, then show that $\boldsymbol{n} \cdot \nabla u(y) > 0$, where $\boldsymbol{n}$ is the outward pointing unit normal at y.