1. Let \( f \) be the function defined by \( f(x) = 1 \) if \( x > 0 \), \( f(0) = 0 \), and \( f(x) = -1 \) if \( x < 0 \). Let \( u \) be a solution of the PDE \( \partial_t u - \frac{1}{2} \partial^2_x u = 0 \) for \( x \in \mathbb{R}, t > 0 \) with initial data \( u(x, 0) = f(x) \). Find a formula for \( u(x, t) \), and express your answer in terms of the error function.

2. Let \( X_n(x) = \sin(\frac{n\pi}{L} x) \), and suppose \( f(x) = B_1 X_1(x) + B_2 X_2(x) + B_3 X_3(x) \) for some constants \( B_1, B_2 \) and \( B_3 \). Compute \( \int_0^L f(x)^2 \, dx \) in terms of \( B_1, B_2, B_3 \) and \( L \).

3. Please only do ONE of the two following subparts. The second subpart is the higher dimensional analogue of the first, and is worth more points. You will only get credit for ONE of these two subparts, so please don’t do them both.
   (a) Let \( L > 0 \), and \( a \) be some given function defined on \([0, L]\). Must we necessarily have
   \[
   \int_0^L [a \partial_x^2 u + (\partial_x a)(\partial_x u)] \, v \, dx = \int_0^L u \left[ a \partial_x^2 v + (\partial_x a) \cdot (\partial_x v) \right] \, dx
   \]
   for all functions \( u, v \) such that \( u(0) = v(0) = u(L) = v(L) = 0 \). Justify your answer. You may assume that the functions \( a, u, \) and \( v \) are infinitely differentiable.
   (b) Let \( D \subseteq \mathbb{R}^2 \) be a bounded domain, and \( a \) be some given function defined on \( D \). Must we necessarily have
   \[
   \int_D [a \triangle u + (\nabla a) \cdot (\nabla u)] \, v \, dx \, dy = \int_D u \left[ a \triangle v + (\nabla a) \cdot (\nabla v) \right] \, dx \, dy
   \]
   for all functions \( u, v \) which are 0 on the boundary of \( D \). Justify your answer. You may assume that the functions \( a, u, \) and \( v \) are as infinitely differentiable.

The next question is a little tricky. While the correct solution is very clean and short, arriving at the solution given what you’ve seen so far isn’t too easy.

4. Let \( f \) be a function such that \( \int_0^L f(x) \, dx < \infty \). Define as usual
   \[
   B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) \, dx.
   \]
   Show that
   \[
   \frac{L}{2} (B_1^2 + B_2^2 + B_3^2) \leq \int_0^L f(x)^2 \, dx.
   \]
   [This is known as Bessel’s inequality. You may not use Parseval’s identity which I stated, but have not yet proved in class.]