880 Stochastic Calculus: Midterm.

Wed 10/19

- *This is a closed book test.*
- *You have 90 mins. The exam has a total of 5 questions and 50 points.*
- *You may use without proof any result that has been proved in class or on the homework.*
- *You may (or may not) find the questions getting progressively harder.*

In this exam, Ω always denotes a probability space, with measure P. Brownian motion will usually be denoted by W or B, and the underlying filtration (if not explicitly mentioned) is denoted by $\mathcal{F} = \{\mathcal{F}_t\}_{t \geqslant 0}$, and is always assumed to satisfy the usual conditions.

[10] 1. Let τ be a stopping time. Show that τ is $\mathcal{F}_\tau$-measurable.

[10] 2. Define the process X to be the Itô integral

$$X_t = \int_0^t \exp\left(\frac{W_s^2}{5s}\right) dW_s.$$

Is X a (continuous) martingale? Prove your answer.

[10] 3. Let $X_t = W_t^3$. Compute $\frac{d}{dt}\langle X \rangle_t$. [By compute, I mean find a deterministic function $f(x,t)$ such that $\frac{d}{dt}\langle X \rangle_t = f(W_t, t)$.]

[10] 4. Let M be a (right) continuous martingale, and let τ be a stopping time. Show that $E(M_t \mid \mathcal{F}_\tau) = M_{\tau \wedge t}$.

[10] 5. Let X be a stochastic process. Suppose there is a function p such that for all $0 < t_1 < \cdots < t_n$, and $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$ we have

$$P((X_{t_1}, \dots, X_{t_n}) \in A_1 \times \cdots \times A_n) = \int_{A_1 \times \cdots \times A_n} p(0, t_1, x_1)\, p(t_1, t_2, x_2 - x_1) \cdots p(t_{n-1}, t_n, x_n - x_{n-1})\, dx_1 \cdots dx_n.$$

for some function p. Show that X has independent increments.