Kolmogorov Forward Equation.

We derived the Kolmogorov backward equation in class. This short note deduces the Kolmogorov forward equation from the Kolmogorov backward equation. Let X be a diffusion satisfying the SDE

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t,$$

where b and σ are time independent and Lipshitz. Suppose further X has a (smooth) transition density

$$p(x,s;y,t) = P(X_t^{(x,s)} \in dy).$$

We know that p satisfies the Kolmogorov backward equation in the initial variables x and s. Namely,

$$\partial_s p + L_x p = 0, \quad \text{and} \quad \lim_{s \to t^-} p(\cdot, s; y, t) = \delta_y,$$

where

$$L = \sum_i b_i \partial_i + \frac{1}{2} \sum_{i,j} a_{i,j} \partial_{i,j}$$

and

$$a_{i,j} = a_{i,j}(x) = \sum_k \sigma_{i,k}(x) \sigma_{j,k}(x).$$

The Kolmogorov forward equation says that p satisfies the dual equation in the variables y, t.

Proposition 1 (Kolmogorov forward equation)**.** *Let L^* be the dual of L, defined by*

$$L^* g = \sum_i -\partial_i(b_i g) + \frac{1}{2} \sum_{i,j} \partial_{i,j}(a_{i,j} g).$$

Then

$$\partial_t p - L_y^* p = 0, \quad \textit{and} \quad \lim_{t \to s^+} p(x, s; \cdot, t) = \delta_x,$$

Remark. The operator L^* is the dual of L with respect to the L^2 inner product. Namely, if $f, g \in C_c^2(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} Lf(x)\, g(x)\, dx = \int_{\mathbb{R}^d} f(x)\, L^* g(x)\, dx$$

Proof. Fix $T > 0$, and always assume $0 \leqslant s < t \leqslant T$. Let $f \in C_b^2(\mathbb{R}^d)$, and define

$$u(x,s) = E^{(x,s)} f(X_T) = \int p(x,s;y,T) f(y)\, dy.$$

We know that u satisfies the Kolmogorov backward equation

$$\partial_s u + L_x u = 0 \quad \text{and} \quad u(x,T) = f(x). \tag{1}$$

Also by the Markov property, it immediately follows that

$$u(x,s) = E^{(x,s)} f(X_t) = \int p(x,s;y,t) u(y,t)\, dy$$

for any $t \in [s,T]$. (Alternately, you can deduce the above equality using uniqueness of solutions to (1).) Now, differentiating both sides in t, and using (1) gives

$$\begin{aligned} 0 = \partial_t u(x,s) &= \int \left[\partial_t p(x,s;y,t) u(y,t) + p(x,s;y,t) \partial_t u(y,t)\right] dy \\ &= \int \left[\partial_t p(x,s;y,t) u(y,t) - p(x,s;y,t) L_y u(y,t)\right] dy \\ &= \int \left[\partial_t p(x,s;y,t) - L_y^* p(x,s;y,t)\right] u(y,t)\, dy \end{aligned}$$

Note, to obtain the last inequality we had to integrate by parts. All the boundary terms involved are 0 because one can show that p vanishes at infinity.

Now choosing $t = T$, we see that

$$\int \left[\partial_t p(x,s;y,T) - L_y^* p(x,s;y,T)\right] f(y)\, dy = 0.$$

Since $f \in C_0^2$ is arbitrary, we are done. $\square$