Kolmogorov Forward Equation.

We derived the Kolmogorov backward equation in class. This short note deduces the Kolmogorov forward equation from the Kolmogorov backward equation. Let $X$ be a diffusion satisfying the SDE

$$dX_t = b(X_t) dt + \sigma(X_t) dW_t,$$

where $b$ and $\sigma$ are time independent and Lipshitz. Suppose further $X$ has a (smooth) transition density $p(x, s; y, t) = P(X_t^{(x,s)} \in dy)$. We know that $p$ satisfies the Kolmogorov backward equation in the initial variables $x$ and $s$. Namely,

$$\partial_s p + L_x p = 0, \quad \text{and} \quad \lim_{s \to t} p(\cdot, s; y, t) = \delta_y,$$

where

$$L = \sum_i b_i \partial_i + \frac{1}{2} \sum_{i,j} a_{i,j} \partial_{i,j}$$

and

$$a_{i,j} = a_{i,j}(x) = \sum_k \sigma_{i,k}(x) \sigma_{j,k}(x).$$

The Kolmogorov forward equation says that $p$ satisfies the dual equation in the variables $y, t$.

**Proposition 1** (Kolmogorov forward equation). Let $L^*$ be the dual of $L$, defined by

$$L^* g = \sum_i -\partial_i (b_i g) + \frac{1}{2} \sum_{i,j} \partial_{i,j} (a_{i,j} g).$$

Then

$$\partial_t p + L_y p = 0, \quad \text{and} \quad \lim_{t \to s^+} p(x, s; y, t) = \delta_x,$$

Remark. The operator $L^*$ is the dual of $L$ with respect to the $L^2$ inner product. Namely, if $f, g \in C^2_c(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} Lf(x) g(x) \, dx = \int_{\mathbb{R}^d} f(x) L^* g(x) \, dx.$$

**Proof.** Fix $T > 0$, and always assume $0 \leq s < t \leq T$. Let $f \in C^2_c(\mathbb{R}^d)$, and define

$$u(x, s) = E^{(x,s)} f(X_T) = \int p(x, s; y, T) f(y) \, dy.$$

We know that $u$ satisfies the Kolmogorov backward equation

$$\partial_s u + L_x u = 0 \quad \text{and} \quad u(x, T) = f(x). \quad (1)$$

Also by the Markov property, it immediately follows that

$$u(x, s) = E^{(x,s)} f(X_t) = \int p(x, s; y, t) u(y, t) \, dy$$

for any $t \in [s, T]$. (Alternately, you can deduce the above equality using uniqueness of solutions to [1].) Now, differentiating both sides in $t$, and using [1] gives

$$0 = \partial_t u(x, s) = \int [\partial_t p(x, s; y, t) u(y, t) + p(x, s; y, t) \partial_t u(y, t)] \, dy$$

$$= \int [\partial_t p(x, s; y, t) u(y, t) - p(x, s; y, t) L_y u(y, t)] \, dy$$

$$= \int [\partial_t p(x, s; y, t) - L^*_y p(x, s; y, t)] u(y, t) \, dy$$

Note, to obtain the last inequality we had to integrate by parts. All the boundary terms involved are 0 because one can show that $p$ vanishes at infinity.

Now choosing $t = T$, we see that

$$\int [\partial_t p(x, s; y, T) - L^*_y p(x, s; y, T)] f(y) \, dy = 0.$$

Since $f \in C^2_0$ is arbitrary, we are done. \(\Box\)