Homework Assignment 5

Assigned Mon 11/28. Due Fri 12/09.

1. *(Ornstein-Uhlenbeck process)* Find an explicit solution of the SDE
$$dX_t = \mu X_t \, dt + \sigma \, dW_t$$
where $\mu, \sigma \in \mathbb{R}$, and W is a 1D Wiener process. Also compute EX_t and $\operatorname{Var}(X_t)$.

2. (a) *(Brownian bridge)* Let $a, b \in \mathbb{R}$, W be a 1D Wiener process. Show that strong existence and uniqueness holds for the SDE
$$dX_t = \frac{b - X_t}{1-t} \, dt + dW_t; \quad t \in [0,1), X_0 = a.$$
Show that $\lim_{t \to 1^-} X_t = b$ almost surely. [This is called a Brownian bridge from a to b.]

   (b) For any $T \in (0,1]$, are the laws of $\{X_t\}_{t \leqslant T}$ and $\{W_t\}_{t \leqslant T}$ absolutely continuous? Prove it. Also, in the case the laws are absolutely continuous, find the Radon Nikodym derivative.

3. *(Weak Uniqueness for Lipschitz coefficients)* Let b, σ satisfy the usual uniform Lipschitz and linear growth conditions. Suppose μ is a probability measure with finite variance. Show that any two (weak or strong) solutions to the SDE
$$dX_t = b_t(X_t) \, dt + \sigma_t(X_t) \, dW_t$$
with initial distribution μ have the same law. [Of course, the assumptions on b, σ guarantee strong uniqueness, and a standard theorem (which we did not prove) guarantees that strong uniqueness implies weak uniqueness. While the general proof that strong uniqueness implies weak uniqueness is a little technical, the above can be done quickly with 'bare hands'.]

4. Let $d \in \mathbb{N}$, $b : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$ be bounded, Borel measurable and $\sigma : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^{d^2}$ be bounded and uniformly Lipschitz. Suppose further there exists $\lambda > 0$ such that for all $t \geqslant 0$ and $x, y \in \mathbb{R}^d$ we have
$$\sum_{i,j,k} \sigma_t^{(i,k)}(x) \sigma_t^{(j,k)}(x) y^{(i)} y^{(j)} = |\sigma_t(x)^* y|^2 \geqslant \lambda |y|^2.$$
Then prove weak existence and uniqueness for the SDE
$$dX_t = b_t(X_t) \, dt + \sigma_t(X_t) dW_t$$
for any given initial distribution μ.

5. Let b, σ be uniformly Lipschitz functions on $\mathbb{R}^d$, and X be the (unique, strong) solution of the SDE $dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t$ with initial data $X_0 = x$.

   (a) Show that $\lim_{t \to 0^+} \frac{1}{t} E(X_t - x) = b(x)$ and $\lim_{t \to 0^+} \frac{1}{t} E(X_t^{(i)} - x^{(i)})(X_t^{(j)} - x^{(j)}) = \sum_k \sigma_{ik}(x) \sigma_{jk}(x)$.

   (b) Show that for all $\varepsilon > 0$, $\lim_{t \to 0^+} \frac{1}{t} P(|X_t - x| > \varepsilon) = 0$.