1. Does there exist a process $Y$ adapted to $\{F^W_t\}_{t \geq 0}$ with $E \int_0^1 Y_s^2 \, ds < \infty$ such that

$$\int_0^1 Y_s \, dW_s = \int_0^1 \left( W_s^2 - \frac{1}{2} \right) \, ds?$$

If yes, find $Y$. If no, prove it.

2. For any $b \in \mathbb{R}$, define $\sigma(b) = \inf \{s \mid W_s - s = b\}$. Show that $\lim_{b \to -\infty} \sigma(b) = \infty$ almost surely.

3. (a) Let $Z_t = \exp \left( \int_0^t 2W_s \, dW_s - \frac{1}{2} \int_0^t 4W_s^2 \, ds \right)$.

Verify that the Kazamaki and Novikov conditions will not directly show that $\{Z_t\}_{t \leq 1} \in \mathcal{M}_c[0, 1]$. [This process is however a martingale, as we will see below.]

(b) Suppose $(t_n) \nearrow \infty$ is sequence of times such that $E \exp \left( \frac{1}{2} \int_{t_n}^{t_{n+1}} |b_s|^2 \, ds \right) < \infty$. Show that $Z_t = \exp \left( \int_0^t b_s \, dW_s - \frac{1}{2} \int_0^t |b_s|^2 \, ds \right)$ is a martingale. [Feel free to use the Novikov condition.]

(c) Suppose there exists a constant $C$ such that $|b_t(x)| \leq C(1 + |x|)$. Show that

$$Z_t = \exp \left( \int_0^t b_s(W_s) \, dW_s - \frac{1}{2} \int_0^t |b_s(W_s)|^2 \, ds \right)$$

is a martingale. Conclude that the process defined in part (a) is in $\mathcal{M}_c[0, 1]$.

4. (Harmonic functions) Let $\{W_t, \mathcal{F}_t\}$ and $\{P^x\}_{x \in \mathbb{R}^d}$ be Brownian family on $\mathbb{R}^d$, $D \subseteq \mathbb{R}^d$ be a bounded domain, and $\tau$ be the exit time of $W$ from $D$. Given $f \in L^\infty(\partial D, \mathcal{B})$, we define

$$u(x) = E^x f(W_\tau) \quad (1)$$

(a) Suppose $\sigma$ is a stopping time such that $\sigma \leq \tau$. Show that

$$u(x) = E^x u(W_\sigma).$$

(b) (Mean value property) Suppose $B(x, R) \subseteq D$, and let $\sigma$ be the exit time of $W$ from $B(x, R)$. Show that

$$u(x) = \frac{1}{|\partial B(x, R)|} \int_{\partial B(x, R)} u(y) \, d\sigma(y) \quad (2)$$

where $\sigma$ denotes the surface measure on $\partial B(x, R)$.

It is a standard PDE result that once (2) is satisfied, $u \in C^2(D)$, and is harmonic (i.e. $\triangle u = 0$). However, without use of the mean value property, we can directly show that $u$ is harmonic provided we assume $u \in C^2(D)$.

(c) If additionally $u \in C^2(D)$, show that $\triangle u = 0$.

(d) Conversely, if $v \in C^2(D) \cap C(\bar{D})$ satisfies $\triangle v = 0$ in $D$, and $v = f$ on $\partial D$, show that $v(x) = E^x f(W_\tau)$. 