Homework Assignment 4

Assigned Thu 11/10. Due Mon 11/28.

1. Does there exist a process Y adapted to $\{\mathcal{F}_t^W\}_{t\geqslant 0}$ with $E\int_0^1 Y_s^2\,ds < \infty$ such that
$$\int_0^1 Y_s\,dW_s = \int_0^1 \left(W_s^2 - \frac{1}{2}\right) ds?$$
If yes, find Y. If no, prove it.

2. For any $b \in \mathbb{R}$, define $\sigma(b) = \inf\{s \mid W_s - s = b\}$. Show that $\lim\limits_{b\to-\infty} \sigma(b) = \infty$ almost surely.

3. (a) Let
$$Z_t = \exp\left(\int_0^t 2W_s\,dW_s - \frac{1}{2}\int_0^t 4W_s^2\,ds\right).$$
Verify that the Kazamaki and Novikov conditions will not directly show that $\{Z_t\}_{t\leqslant 1} \in \mathcal{M}_c[0,1]$. [This process is however a martingale, as we will see below.]

   (b) Suppose $(t_n) \nearrow \infty$ is sequence of times such that $E\exp\left(\frac{1}{2}\int_{t_n}^{t_{n+1}} |b_s|^2\,ds\right) < \infty$. Show that $Z_t = \exp\left(\int_0^t b_s\,dW_s - \frac{1}{2}\int_0^t |b_s|^2\,ds\right)$ is a martingale. [Feel free to use the Novikov condition.]

   (c) Suppose there exists a constant C such that $|b_t(x)| \leqslant C(1+|x|)$. Show that
$$Z_t = \exp\left(\int_0^t b_s(W_s)\,dW_s - \frac{1}{2}\int_0^t |b_s(W_s)|^2\,ds\right)$$
is a martingale. Conclude that the process defined in part (a) is in $\mathcal{M}_c[0,1]$.

4. *(Harmonic functions)* Let $\{W_t, \mathcal{F}_t\}$ and $\{P^x\}_{x\in\mathbb{R}^d}$ be Brownian family on $\mathbb{R}^d$, $D \subseteq \mathbb{R}^d$ be a bounded domain, and τ be the exit time of W from D. Given $f \in L^\infty(\partial D, \mathcal{B})$, we define
$$u(x) = E^x f(W_\tau) \tag{1}$$

   (a) Suppose σ is a stopping time such that $\sigma \leqslant \tau$. Show that
$$u(x) = E^x u(W_\sigma).$$

   (b) *(Mean value property)* Suppose $B(x,R) \subseteq D$, and let σ be the exit time of W from $B(x,R)$. Show that
$$u(x) = \frac{1}{|\partial B(x,R)|}\int_{B(x,R)} u(y)\,d\sigma(y) \tag{2}$$
where σ denotes the surface measure on $\partial B(x,R)$.

   It is a standard PDE result that once (2) is satisfied, $u \in C^2(D)$, and is harmonic (i.e. $\triangle u = 0$). However, without use of the mean value property, we can directly show that u is harmonic provided we assume $u \in C^2(D)$.

   (c) If additionally $u \in C^2(D)$, show that $\triangle u = 0$.

   (d) Conversely, if $v \in C^2(D) \cap C(\bar{D})$ satisfies $\triangle v = 0$ in D, and $v = f$ on ∂D, show that $v(x) = E^x f(W_\tau)$.