Homework Assignment 3

Assigned Thu 10/27. Due Wed 11/09.

1. Let $\mathcal{F}$ denote the Borel σ-algebra on the Wiener space $C[0,\infty)^d$, and let P be the Wiener measure. As we did in class, for any $x \in \mathbb{R}$, $F \in \mathcal{F}$, define $P^x(F) = P(F - x)$. Show that the function $x \mapsto P^x(F)$ is Borel measurable.

2. Let $\Omega = C[0,\infty)^d$, and X be the canonical coordinate mapping process (i.e. $X_t(\omega) = \omega(t)$ for all $\omega \in \Omega$). For a fixed $t \geqslant 0$, define the *time shift* operator $\theta_t : \Omega \to \Omega$ by $\theta_t(\omega) = \omega(t + \cdot)$. That is $\theta_t\omega(s) = \omega(s+t)$. If P is a probability measure on Ω under which X is a Markov process (under the canonical filtration), then show that $P(\theta_t F \mid \mathcal{F}_s) = P(\theta_t F \mid X_s)$.

3. (a) Let B be a Brownian motion. Find a Borel function f such that the process $\{f(B_t)\}_{t\geqslant 0}$ is not a Markov process.
   (b) Show that the process $X_t = |B_t|$ is a Markov process. Further show that the transition density $p_+(h,x,y) \overset{\text{def}}{=} P^0(X_{t+h} \in dy \mid X_t = x) = p(h,x,y) + p(h,x,-y)$, where p is the transition density of Brownian motion, and $x, y \geqslant 0$.
   (c) Let $Y_t = M_t - B_t$, where $M_t = \sup_{s\leqslant t} B_s$. Show that Y is a Markov process, with the same transition density as X. Conclude that X and Y have the same finite dimensional distributions.

4. *(Tanaka's formula and local time)* Let W be a standard 1D Brownian motion, and define
$$L_t^\varepsilon = \frac{1}{2\varepsilon}\lambda\left\{s \in [0,t] \;\middle|\; |W_s| \leqslant \varepsilon\right\}$$
where λ denotes the Lebesgue measure. One would naturally expect that $\lim\limits_{\varepsilon\to 0^+} L_t^\varepsilon$ measures the amount of time Brownian motion spends at 0. This problem proves the existence of limit.
   (a) Let f_ε be the (unique) function such that $f_\varepsilon(0) = f_\varepsilon'(0) = 0$, and $f_\varepsilon'' = \frac{1}{\varepsilon}\chi_{[-\varepsilon,\varepsilon]}$. Show that
$$f_\varepsilon(W_t) - f_\varepsilon(0) = \int_0^t f_\varepsilon'(W_s)\,dW_s + L_t^\varepsilon$$
   (b) As $\varepsilon \to 0$, show that $E\int_0^t |f_\varepsilon'(W_s) - \operatorname{sign}(W_s)|^2\,ds \to 0$.
   (c) As $\varepsilon \to 0$, show that $f_\varepsilon(0) \to 0$, and $E|f_\varepsilon(W_t) - |W_t||^2 \to 0$.
   (d) Conclude there exists an adapted, square integrable process L such that $E|L_t^\varepsilon - L_t|^2 \to 0$ as $\varepsilon \to 0$. Further show
$$|W_t| = \int_0^t \operatorname{sign}(W_s)\,dW_s + L_t$$
   This is called Tanaka's formula. [Remark: If we set $f(x) = |x|$, then $f'(x) = \operatorname{sign}(x)$, and $f''(x) = 2\delta_0$. Thus if we formally apply Itô's formula to $f(W)$, we exactly arrive at Tanaka's formula. Of course, since $f \notin C^2$, we may not apply Itô, and thus we have to resort to the approximations outlined above.]

5. *(Bessel processes)* Let $d > 1$, and W be a standard d-dimensional Brownian motion. Let $R = |W|$.
   (a) Let $B = \sum_1^d \int_0^t \frac{W_s^{(i)}}{R_s}\,dW_s^{(i)}$. Show that B is a standard 1D Brownian motion.
   (b) Show that $dR_t = \dfrac{d-1}{2R_t}\,dt + dB_t$. [Remark: Tanaka's formula shows that for $d = 1$, this equation does not hold. The Bessel process can be used to study questions about the return of Brownian motion to the origin. We know for $d = 1$, $P(|W_t| > 0\ \forall t > 0\} = 0$. However, for $d > 1$, $P(|W_t| > 0\ \forall t > 0) = 1$.
   It turns out that for $d = 2$, while Brownian motion does not return to the origin, it does come arbitrarily close. However, for dimensions 3 and higher, once Brownian motion leaves the origin, it remains bounded away from the origin almost surely. Details of the proof of this can be found in Karatzas and Shreve §3.3C]