Homework Assignment 2
Assigned Thu 10/06. Due Wed 10/26.

1. Let $B$ be a standard 1D Brownian motion. Show that each of the following are also standard 1D Brownian motions:
   (a) $\{-B_t\}_{t \geq 0}$
   (b) $\{\frac{1}{\sqrt{\lambda}} B_{\lambda t}\}_{t \geq 0}$, for any $\lambda > 0$.
   (c) $W_t = tB_{\frac{1}{t}}$ for $t > 0$, and $W_0 = 0$.
   (d) $\{B_{s+t} - B_s\}_{t \geq 0}$ for any fixed $s \geq 0$.

2. Say $X_t$ is a process with independent increments. Show that if $s < t$, $X_t - X_s$ is independent of $\mathcal{F}_s^X$. [Recall $\mathcal{F}_s^X = \sigma(\cup_{t \leq s} \sigma(X_u))$. Also, $X$ has independent increments means that for any finite sequence $0 = t_0 < t_1 \cdots < t_n$, the family of random variables $\{X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}\}$ is independent.]

3. Suppose $X$ is process with continuous trajectories and independent, stationary increments. (Recall, a process has stationary increments if the distribution of the increment $X_{t+h} - X_t$ only depends on $h$, and not on $t$.) Assume (for normalization) that $X_0 = 0 = EX_1$ and $EX_1^2 = 1$. If further $EX_1^4 < \infty$, then show that $X \in \mathcal{M}_{\infty}^2$, and $\langle X \rangle_t = t$. [Thus if you know Lévy’s criterion, $X$ is a standard Brownian motion.]

4. For any $\lambda > 0$, we define the Poisson process with intensity $\lambda$ as follows. Let $\tau_i$ be a sequence of i.i.d exponential random variables with parameter $\lambda$ (i.e. $P(\tau_i \in dt) = \frac{1}{\lambda} e^{-\lambda t}$). Let $\sigma_n = \sum_{i=1}^n \tau_i$. The intuition is that $\tau_i$ is the time at which the $i^{th}$ customer arrives, and $\sigma_n$ is the time it takes for the first $n$ customers to arrive.
   Now we define $N_t = \max\{n \in \mathbb{N} \mid \sigma_n \leq t\}$ to be the number of customers who’ve arrived up to time $t$. This is called the Poisson process with intensity $\lambda$.
   (a) Show that $N_t - N_s$ is an integer valued Poisson random variable with parameter $\lambda(t-s)$. Further, show $N_t - N_s$ is independent of $\mathcal{F}_s^N$. [Thus $N$ is a process with stationary, independent increments, that is not a Brownian motion. This is a counter example to the previous example, if the continuity in time assumption is dropped.]
   (b) Show that $\{(N_t - \lambda t)^2 - \lambda t\} \in \mathcal{M}_2$. [This is the counter example to Lévy’s criterion, if the continuity in time assumption is dropped.]

5. Let $M \in \mathcal{M}_{\infty}^2$, have absolutely continuous quadratic variation, and $X \in \mathcal{L}(M)$. Define $N = I(X, M)$. If $Y \in \mathcal{L}(N)$, then show that $I(Y, N) = I(Y, M)$.

6. Let $M \in \mathcal{M}_{\infty}^2$ have absolutely continuous quadratic variation. Suppose $X \in \mathcal{L}(M)$, and $\tau$ is a stopping time. Let $\bar{X}$ be defined by $\bar{X}_t = \chi_{\{t \leq \tau\}} X_t$. Show that $\bar{X} \in \mathcal{L}(M)$, $X \in \mathcal{L}(M^\tau)$, and $I(\bar{X}, M) = I(X, M^\tau) = I(X, M)^\tau$. 