Homework Assignment 2

Assigned Thu 10/06. Due Wed 10/26.

1. Let B be a standard 1D Brownian motion. Show that each of the following are also standard 1D Brownian motions:
   (a) $\{-B_t\}_{t\geqslant 0}$
   (b) $\{\frac{1}{\sqrt{\lambda}} B_{\lambda t}\}_{t\geqslant 0}$, for any $\lambda > 0$.
   (c) $W_t = tB_{\frac{1}{t}}$ for $t > 0$, and $W_0 = 0$.
   (d) $\{B_{s+t} - B_s\}_{t\geqslant 0}$ for any fixed $s \geqslant 0$.
2. Say X_t is a process with independent increments. Show that if $s < t$, $X_t - X_s$ is independent of $\mathcal{F}^X_s$. [Recall $\mathcal{F}^X_t = \sigma(\cup_{s\leqslant t}\sigma(X_s))$. Also, X has independent increments means that for any finite sequence $0 = t_0 < t_1 \cdots < t_n$, the family of random variables $\{X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}\}$ is independent.]
3. Suppose X is process with continuous trajectories and independent, stationary increments. (Recall, a process has stationary increments if the distribution of the increment $X_{t+h} - X_t$ only depends on h, and not on t.) Assume (for normalization) that $X_0 = 0 = EX_1$ and $EX_1^2 = 1$. If further $EX_1^4 < \infty$, then show that $X \in \mathcal{M}^2_c$, and $\langle X\rangle_t = t$. [Thus if you know Lévy's criterion, X is a standard Brownian motion.]
4. For any $\lambda > 0$, we define the *Poisson process with intensity* λ as follows. Let τ_i be a sequence of i.i.d exponential random variables with parameter λ (i.e. $P(\tau_i \in dt) = \frac{1}{\lambda}e^{-\lambda t}$). Let $\sigma_n = \sum_1^n \tau_i$. The intuition is that τ_i is the time at which the i^{th} customer arrives, and σ_n is the time it takes for the first n customers to arrive.

   Now we define $N_t = \max\{n \in \mathbb{N} \mid \sigma_n \leqslant t\}$ to be the number of customers who've arrived up to time t. This is called the Poisson process with intensity λ.
   (a) Show that $N_t - N_s$ is an integer valued Poisson random variable with parameter $\lambda(t-s)$. Further, show $N_t - N_s$ is independent of $\mathcal{F}^N_s$. [Thus N is a process with stationary, independent increments, that is *not* a Brownian motion. This is a counter example to the previous example, if the continuity in time assumption is dropped.]
   (b) Show that $\{(N_t - \lambda t)^2 - \lambda t\} \in \mathcal{M}^2$. [This is the counter example to Lévy's criterion, if the continuity in time assumption is dropped.]
5. Let $M \in \mathcal{M}^2_c$, have absolutely continuous quadratic variation, and $X \in \mathcal{L}(M)$. Define $N = I(X, M)$. If $Y \in \mathcal{L}(N)$, then show that $I(Y, N) = I(XY, M)$.
6. Let $M \in \mathcal{M}^2_c$ have absolutely continuous quadratic variation. Suppose $X \in \mathcal{L}(M)$, and τ is a stopping time. Let $\tilde{X}$ be defined by $\tilde{X}_t = \chi_{\{t\leqslant\tau\}} X_t$. Show that $\tilde{X} \in \mathcal{L}(M)$, $X \in \mathcal{L}(M^\tau)$, and $I(\tilde{X}, M) = I(X, M^\tau) = I(X, M)^\tau$.