Homework Assignment 1

Assigned Mon 09/12. Due Mon 10/03.

1. Let σ, τ be two stopping times.
   (a) Show that $\sigma \wedge \tau$, $\sigma \vee \tau$, $\sigma + \tau$ are also stopping times.
   (b) If $\sigma \leqslant \tau$ almost surely, then show $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.

2. (a) Let $\{\mathcal{F}_n \mid n \in \mathbb{N}\}$ be a *decreasing* sequence of σ-algebras (i.e. $\mathcal{F}_n \supseteq \mathcal{F}_{n+1}$), and $\{X_n, \mathcal{F}_n \mid n \in \mathbb{N}\}$ be a backward submartingale (i.e. $E(X_n \,|\, \mathcal{F}_{n+1}) \geqslant X_{n+1}$). If $\inf_{n\in\mathbb{N}} EX_n > -\infty$, then show that $\{X_n \mid n \in \mathbb{N}\}$ is uniformly integrable.
   (b) Let $\{X_t, \mathcal{F}_t\}$ be a right continuous submartingale. Show that the function $t \mapsto EX_t$ is right continuous.

3. Let $\{X_t, \mathcal{F}_t \mid 0 \leqslant t < \infty\}$ be a right continuous martingale. Show that the following are equivalent.
   (a) $\{X_t \mid 0 \leqslant t \leqslant \infty\}$ is a uniformly integrable family of random variables.
   (b) There exists $X_\infty \in L^1(\Omega, \mathcal{F}_\infty)$ such that $X_t \to X_\infty$ in L^1 as $t \to \infty$. (Recall $\mathcal{F}_\infty = \sigma(\cup_t \mathcal{F}_t)$.)
   (c) There exists $X_\infty \in L^1(\Omega, \mathcal{F}_\infty)$ such that $X_t \to X_\infty$ almost surely and $\{X_t, \mathcal{F}_t \mid 0 \leqslant t \leqslant \infty\}$ is a martingale.
   (d) There exists $X_\infty \in L^1(\Omega, \mathcal{F}_\infty)$ such that $\{X_t, \mathcal{F}_t \mid 0 \leqslant t \leqslant \infty\}$ is a martingale.

4. Let X be an integrable, progressively measurable process such that $EX_\tau = 0$ for all stopping times τ such that $P(\tau < \infty) = 1$. Show that X is a martingale.

5. Let X be a continuous local martingale. Show that there exists a localizing sequence (τ_n) such that for all n, the stopped process $X^{\tau_n} = \{X_{\tau_n \wedge t}, \mathcal{F}_t\}$ is bounded almost surely. (Note, we say (τ_n) is a localising sequence if it is an increasing sequence of stopping times which converges to ∞ almost surely, and for all n, the stopped process X^{τ_n} is a martingale).

6. Given any process X, and $p \in [1, \infty)$ define the p^{th} variation of X to be the process

$$V^p(X)_t = \lim_{|\Delta_n| \to 0} \sum_k \left| X_{t_{k+1} \wedge t} - X_{t_k \wedge t} \right|,$$

   provided for each t the limit exists in probability, and is independent of the sequence of partitions (Δ_n). (As usual $\Delta_n = \{0 = t_0 < \cdots < t_N = T\}$ is a partition of $[0, T]$, and we omit the dependence of t_k on n for convenience.) If $X \in \mathcal{M}_c^2$, show that $V^p(X)$ exists for all $p \in [1, \infty)$. Further, show $V^p(X) = 0$ for all $p < 2$ and $V^p(X) = \infty$ for all $p > 2$.

7. Let $M = \{M_t, \mathcal{F}_t \mid t \in [0, T)\}$ be a square integrable, continuous martingale with $M_0 = 0$ almost surely. Let $\langle M \rangle_{T^-}$ denote $\lim_{t \to T^-} \langle M \rangle_t$ (which exists since $\langle M \rangle$ is an increasing function of time). Show that

$$P\left(\{\langle M \rangle_{T^-} < \infty\} \Delta \left\{ \lim_{t \to T^-} M_t \text{ exists, and is finite} \right\} \right) = 0$$

$$\text{and} \qquad P\left(\{\langle M \rangle_{T^-} = \infty\} \Delta \left\{ \varlimsup_{t \to T^-} |M_t| = \infty \right\} \right) = 0$$

   Here $A \Delta B = A \cup B - A \cap B$ denotes the symmetric difference between A and B. [This is the moral equivalent of the statement $\lim_{t \to T^-} M_t$ exists and is finite "if and only if" $\lim_{t \to T^-} \langle M \rangle_t$ exists and is finite.]