880 Stochastic Calculus: Final.

Dec 15$^{\text{th}}$, 2011

- *This is a closed book test.*
- *You have 3 hours. The exam has a total of 7 questions and 30 points.*
- *You may use without proof any result that has been proved in class or on the homework.*
- *You may (or may not) find the questions getting progressively harder.*

[4] 1. Let W be a standard 1D Brownian motion. Compute $\int_0^t \sin(W_s)\,dW_s$, and express your answer in the form $\int_0^t f(W_s,s)\,ds + g(W_t,t)$ for some explicit, deterministic functions f and g.

[4] 2. Let W be a standard 1D Brownian motion, and $f, g \in \mathcal{L}(W)$. Define the process X by

$$X_t = \int_0^t f_s\,ds + \int_0^t g_s\,dW_s.$$

Find a necessary and sufficient criterion for f and g which guarantees that X is a standard 1D Brownian motion. Prove your answer.

[4] 3. Let W be a two dimensional Brownian motion starting from the point $(1,0) \in \mathbb{R}^2$. Let $X_t = \ln|W_t|$, $\tau_n = \inf\{t \mid |W_t| \leqslant 1/n\}$ be the hitting time of W to the closed ball $\bar{B}(0,1/n)$ [remember that W starts from $(1,0)$, *outside* this ball]. For all $n \in \mathbb{N}$, is X^{τ_n} a continuous martingale? Prove it. [Recall X^{τ_n} is the stopped process is defined by $X_t^{\tau_n} = X_{\tau_n \wedge t}$. Further, when addressing whether X^{τ_n} or not, the filtration in question is always the (augmented) Brownian filtration.]

4. Let $x \in \mathbb{R}^2$ be non-zero, and W be a two dimensional Brownian motion starting from x.

[4] (a) Let $R > |x|$ be fixed. For every $n \in \mathbb{N}$ define

$$\tau_{n,R} = \inf\{t \geqslant 0 \bigm| W_t \notin B_R - \overline{B_{1/n}}\} = \inf\{t \geqslant 0 \bigm| |W_t| \notin (1/n, R)\}$$

Compute $\displaystyle\lim_{n\to\infty} P\left(\left|W_{\tau_{n,R}}\right| = R\right)$.

[2] (b) Let $\tau = \inf\{t \geqslant 0 \mid W_t = 0\}$. Compute $P(\tau = \infty)$.

[4] 5. Let W, X be as in Question 3. Is X a martingale? Prove it. [Again, the filtration in question is the (augmented) Brownian filtration.]

[4] 6. Let M be a continuous martingale, and suppose $EM_t^2 \leqslant c(1+t^\beta)$ for some $c, \beta > 0$. For any $\alpha > \frac{\beta}{2}$ show that $\displaystyle\lim_{t\to\infty} \frac{|M_t|}{t^\alpha}$ exists almost surely, and compute it's value.

[4] 7. Let X be a d-dimensional diffusion satisfying the SDE

$$dX_t = b(X_t)\,dt + \sigma(X_t)dW_t,$$

where b and σ are time independent and Lipshitz. Let $D \subseteq \mathbb{R}^d$ be a domain, and τ be the exit time of X from D. Suppose

$$u \in C^{2,1}(D \times (0,\infty)) \cap C(\bar{D} \times (0,\infty)) \cap C(D \times [0,\infty))$$

satisfies

$$\partial_t u - Lu = 0 \quad \text{in } D, \qquad u(x,0) = 1 \quad \text{in } D, \qquad u(x,t) = 0 \quad \text{on } \partial D \times (0,\infty),$$

where $L = \sum_i b_i \partial_i + \frac{1}{2}\sum_{i,j} a_{i,j}\partial_i\partial_j$, and $a_{i,j} = \sum_k \sigma_{i,k}\sigma_{j,k}$. Show that $u(x,t) = P^x(\tau \geqslant t)$. [Note: The maximum principle guarantees $0 \leqslant u(x,t) \leqslant 1$, which you may use in your solution.]