Math 880 Final.

December 14, 2010

This is a closed book test. You have 3 hours. You may (or may not) find the questions getting progressively harder. You may use without proof any result from class and homework.

In this exam, Ω always denotes a probability space, with measure P. Brownian motion will usually be denoted by W or B, and the underlying filtration (if not explicitly mentioned) is denoted by $\mathcal{F} = \{\mathcal{F}_t\}_{t \geqslant 0}$, and is always assumed to satisfy the usual conditions.

1. Compute $E \frac{d}{dt} \langle W^2 \rangle_t$ explicitly as a function of t.

2. Let $X_t = 2t + W_t$, and $Y_t = t + 2W_t$. Let P^X and P^Y be the laws of X and Y respectively. Are they absolutely continuous with respect to each other? Prove it. If your answer is yes, find $\frac{dP^X}{dP^Y}$.

3. Let $T > 0$ be fixed. Does there exist a process Y adapted to $\{\mathcal{F}_t^W\}_{t \geqslant 0}$ with $E \int_0^T Y_s^2 \, ds < \infty$ such that
$$\int_0^T Y_s \, dW_s = \int_0^T W_s^2 \, ds - \frac{T^2}{2}?$$
If yes, find Y. If no, prove it.

4. Suppose X is the solution of the SDE $dX_t = X_t \, dt + \frac{X_t^2}{1+X_t^2} \, dW_t$ with $X_0 \sim N(0,1)$. Compute EX_t explicitly as a function of t.

5. Let $\{(M_t, \mathcal{F}_t) \mid 0 \leqslant t \leqslant \infty\}$ be a right continuous martingale (with last element). Suppose for some $p \in (1, \infty)$ we have $E|M_\infty|^p < \infty$. Show that $(M_t) \to M_\infty$ in L^p (i.e. show $\lim_{t \to \infty} E|M_t - M_\infty|^p = 0$).

6. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a bounded, Lipschitz function, and let X be the diffusion with drift b and diffusion coefficient I (i.e. $X_t^x = x + \int_0^t b(X_s^x) \, ds + W_t$, where W is a standard d-dimensional Brownian motion).
   (a) Let $A \in \mathcal{B}(\mathbb{R}^d)$ have non zero Lebesgue measure. Prove $P(X_t^x \in A) > 0$ for any $x \in \mathbb{R}^d$, $t > 0$.
   (b) *(Strong maximum principle)* Let $f : \mathbb{R}^d \to \mathbb{R}$, be Borel measurable and bounded. Define $u(x,t) = Ef(X_t^x)$. If f is not (almost everywhere) constant, show that for any $x \in \mathbb{R}^d$, $t > 0$,
   $$u(x,t) < \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} f(x).$$
   Recall $\operatorname{ess\,sup} f = \inf\{\lambda \in \mathbb{R} \mid |\{f > \lambda\}| = 0\}$, where $|\{f > \lambda\}|$ denotes the Lebesgue measure of the set $\{x \mid f(x) > \lambda\}$.

7. Let $W^{(1)}$ and $W^{(2)}$ be two independent Brownian motions. Define X to be the process
$$X_t = \int_0^t W_s^{(2)} \, ds + W_t^{(1)}$$
and let $\{\mathcal{F}_t^X\}_{t \geqslant 0}$ denote the filtration generated by the process X. Let Y be defined by
$$Y_t = \int_0^t \left(W_s^{(2)} - E\left(W_s^{(2)} \;\middle|\; \mathcal{F}_s^X \right) \right) ds + W_t^{(1)}$$
Is Y a Brownian motion with respect to it's own filtration. Prove or disprove?