Math 372 Syllabus and Lecture Schedule.

Gautam Iyer, Spring 2012

Lecture 1, Mon 1/16: Introduction and motivation.

- *(Sec. 1.1)* Introduction.
  - A PDE is a differential equation involving derivatives with respect to more than one variable.
  - Ubiquitous in nature
    - Heat equation: $\partial_t u - \triangle u = 0$ governs evolution of temperature in a conductor.
    - Wave equation: $\partial_t^2 u - \triangle u = 0$ governs propagation of waves.
    - Laplace equation: $-\triangle u = 0$, satisfied by the stream function of an incompressible fluid. (Such functions are called Harmonic functions.) Steady states of the unforced heat/wave equations.
    - Poisson equation: $-\triangle u = 1$ is the electrostatic potential distribution in a conductor with uniform charge density. Steady states of the forced heat/wave equation.
  - No simple solution formula.
    - Linear ODE's have explicit solutions. Linear PDE's usually do not.
    - Elementary existence theorem for general ODE's. Analogue for PDE's involves the assumption of analyticity. Counter examples exist if this assumption is relaxed.
- *(Sec. 1.2)* Method of characteristics.
  - Consider first $a\partial_x u + b\partial_x u = 0$, where a, b are functions of x, y.
    - $\binom{a}{b} \cdot \nabla u = 0$ means the vector $\binom{a}{b}$ is tangential to level sets of u.

Lecture 2, Wed 1/18: Method of characteristics.

- Solve the ODE $\frac{dx}{a} = \frac{dy}{b}$ to find the level sets of u (called *characteristic curves*).
- By solving the above ODE, find a function F so that $F(x, y) = c$ describes all characteristic curves.
- $u(x, y) = f(F(x, y))$ for an arbitrary f is the general solution.

- E.g. general solution of $-y\partial_x u + x\partial_y u = 0$ is $u = f(x^2 + y^2)$. Characteristics are circles with center the origin.
- For $3D$, same trick works. Consider the PDE $au_x + bu_y + cu_z = 0$
  - Characteristics are given by the ODE $\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c}$.
  - Solve these ODE's, and write characteristics as in the form $F(x, y, z) = c_1$, and $G(x, y, z) = c_2$ (where F, G are functions you should explicitly find), and c_1, c_2 are constants.
  - The general solution is of the form $u(x, y, z) = f(F(x, y, z), G(x, y, z))$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a general (differentiable) function.
- E.g. general solution of $u_x + 2u_y + 3u_z = 0$ is of the form $u(x, y, z) = f(2x - y, 3x - z)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a general (differentiable) function.

Lecture 3, Fri 01/20: Derivation of PDE's from physical principles.

- Solving first order, linear *inhomogeneous* equations by the method of characteristics. (See the handout for details.)

- *(Sec. 1.3)* Derivation of PDE's from Physics.
  - One dimensional heat equation.
    - Heat is proportional to temperature.
    - Rate of heat flow is proportional to the temperature gradient.
    - Show $\frac{d}{dt} \int_a^b \theta(x, t) c\rho \, dx = \alpha \int_a^b \partial_x^2 \theta(x, t) \, dx$.

Lecture 4, Mon 1/23.

- Conclude $\partial_t \theta = \kappa \partial_x^2 \theta$.

- Higher dimensional heat equation
  - State the Divergence theorem.
  - Above reasoning shows $\partial_t \theta = \kappa \triangle \theta$.
- One dimensional transport equation.
  - u – concentration of a pollutant in a (1D) flowing pipe.
  - c – velocity of water flow (constant).
  - Show $\frac{d}{dt} \int_a^b u(x, t) \, dx = cu(a, t) - cu(b, t) = -\int_a^b \partial_x u(x, t) \, dx$.

Lecture 5, Wed 01/25.

- Conclude $\partial_t u + c\partial_x u = 0$.
- MOC: $u(x, t) = f(x - ct)$.

- One dimensional wave equation.
  - u – displacement of a particle from it's mean position.
  - T – Tension in the string
  - Conclude $\int_a^b \rho \partial_t^2 u(x, t) \, dx = T(u_x(b, t) - u_x(a, t))$, and hence $\partial_t^2 u = c^2 \partial_x^2 u$, for $c = \sqrt{\frac{\rho}{T}}$.
  - Higher dimensional wave equation: $\partial_t^2 u - \triangle u = 0$.

Lecture 6, Fri 01/27: Boundary conditions.

- *(Sec. 1.4)* Auxiliary condition: Specify $u(x, y)$ along some curve.
  - E.g. Solve $\partial_x u + y\partial_y u = u^2$, subject to the condition $u(0, y) = f(y)$.
    - Characteristics: $y = c_1 e^x$. General solution $u(x, y) = -1/(x + g(ye^{-x}))$.
    - Auxiliary condition gives $g(y) = -1/f(y)$, and substitute back.
- Boundary conditions.
  - Dirichlet: Specify u on the boundary of the domain.

- * Heat equation: $u = 0$ on ∂D corresponds to holding the temperature of the boundary of the conductor constant (e.g. by immersing it in a 'bath' of constant temperature).
  * Wave equation: $u = 0$ on the boundary corresponds to holding the endpoints of a guitar string fixed.
  * Poisson equation: Specifying u on the boundary corresponds to specifying the voltage at the boundary.
- Neumann: Specify $\frac{\partial u}{\partial \hat{n}}$ (the *normal* derivative) on the boundary of the domain.
  * Specifying the full derivative on ∂D is 'too much'. Leads to inconsistencies (proof later).
  * Heat equation: $\frac{\partial u}{\partial n} = 0$ on ∂D corresponds to perfectly insulated boundaries (no heat exchanged with the surroundings).
  * Wave equation: $\frac{\partial u}{\partial n} = 0$ on ∂D corresponds to 'no stress' on the boundary. E.g. allowing one end of the string to move freely in a track perpendicular to the string.
  * Poisson equation: Specifying $\frac{\partial u}{\partial n}$ on ∂D corresponds to specifying the current at the boundary.

- Initial conditions: Specify conditions at time 0.
  - Heat equation: Enough to specify $u(x, 0)$ (initial temperature).
  - Wave equation: Must specify both $u(x, 0)$ (initial position), and $\partial_t u(x, 0)$ (initial velocity).

Lecture 7, Mon 1/30: One dimensional wave equation.

- *(Sec. 2.1)* Wave equation on the line
  - Say $\partial_t^2 u - c^2 \partial_x^2 u = 0$, for $x \in \mathbb{R}$, $t > 0$.
  - Let $v = (\partial_t + c\partial_x)u$. Then $(\partial_t - c\partial_x)v = 0$.
  - M.O.C shows $v(x, t) = h(x + ct)$.
  - By M.O.C, deduce $u(x, t) = f(x - ct) + g(x + ct)$.
  - D'Alembert's principle: If $u(x, 0) = \varphi(x)$ and $\partial_t u(x, 0) = \psi(x)$, then $u(x, t) = \frac{1}{2}(\varphi(x + ct) + \varphi(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi$.

Lecture 8, Wed 02/01: Conservation of Energy and Causality.

- *(Sec. 2.2)* Causality for the 1D wave equation.
  - Domain of dependence is the inverted cone, bounded by particles travelling with speed $\pm c$.
  - Domain of influence is the upright cone, bounded by particles travelling with speed $\pm c$.
  - No information propagates faster than the speed c.
- *(Sec. 2.2)* Conservation of energy.
  - The energy $E = \int_{-\infty}^{\infty} (\partial_t u)^2 + c^2(\partial_x u)^2$ is constant in time.
  - Uniqueness for initial value problem.
    * Say u_1, u_2 are solutions to the forced wave equation $\partial_t^2 u - c^2 \partial_x^2 u = f$, with the same initial position and initial velocity, and decay sufficiently at ∞.
    * Set $v = u_1 - u_2$. Then v is a solution to $\partial_t^2 v - c^2 \partial_x^2 v = 0$.
    * Conservation of energy implies $\int_{-\infty}^{\infty} (\partial_t v)^2 + (\partial_x v)^2$ is constant in time, and hence must be 0 (the energy at time 0).
    * Thus $\partial_x v = \partial_t v = 0$, and so v is constant in both space and time.
    * Since $v = 0$ at time 0, $v = 0$ for all time. Consequently $u_1 = u_2$ as desired.

Lecture 9, Fri 02/03: Maximum principle.

- *(Sec. 2.3)* 1D Heat equation.
  - Physical meaning: "Heat does not collect at hot points".
  - Lemma: Let $R = (0, L) \times (0, T)$. If $\partial_t v - \kappa \partial_x^2 v < 0$, then v does not attain a maximum in the interior of R, or on the top of R.
    * At an interior maximum, $\partial_t v = 0$ and $\partial_x^2 v \leqslant 0$, which gives a contradiction.
  - (Weak) Maximum principle: If $\partial_t u - \kappa \partial_x^2 u \leqslant 0$, then v attains a maximum on the sides or bottom of R. (The maximum, however, can also be attained at interior points of R or on the top of R.)
    * Proof: Let $v(x, t) = u(x, t) + \varepsilon x^2$. Verify $\partial_t v - \kappa \partial_x^2 v < 0$, apply the lemma, and send $\varepsilon \to 0$.
  - Note: The strong Maximum principle says that if v attains it's maximum at an interior point (x_0, t_0), then v must be constant up to time t_0. The proof is much harder, however, is still accessible to you!

Lecture 10, Mon 02/06.

- Maximum principle in higher dimensions. (Complete statement and proof on your HW.)

- Uniqueness for the Dirichlet problem, initial value problem: If u_1, u_2 are solutions to the forced heat equation $\partial_t u - \kappa \triangle u = 0$ in D, with initial conditions $u(x, 0) = \varphi(x)$, and Dirichlet boundary conditions $u(x, t) = g(x, t)$ for $x \in \partial D$. Then $u_1 = u_2$.
  - Maximum principle proof: Set $v = u_1 - u_2$.
  - By the maximum principle, both the maximum and minimum of v are attained either at $t = 0$, or when $x \in \partial D$.
  - By the given initial and boundary conditions, $v = 0$ both when $t = 0$ and $x \in \partial D$. Consequently, by the maximum principle, $v = 0$ identically.

Lecture 11, Wed 02/08.

- Energy decay for the heat equation.
  - Let u satisfy the heat equation with 0 Neumann (or 0 Dirichlet) B.C. Let $E(t) = \int_0^L u(x, t)^2 \, dx$. Then $\frac{d}{dt} E \leqslant 0$.
    * 1D case: $\frac{dE}{dt} = 2 \int_0^L u \partial_t u \, dx = 2\kappa \int_0^L u \partial_x^2 u = 2\kappa \int_0^L (\partial_x u)^2 \leqslant 0$.

- Higher dimensional case: Observe first $\int_D u\triangle u\,dV = \int_D u\nabla\cdot\nabla u\,dV = \int_D \left[\nabla\cdot(u\nabla u) - |\nabla u|^2\right]\,dV = -\int_D |\nabla u|^2\,dV + \int_{\partial D} u\frac{\partial u}{\partial n}\,dS$.
- The last boundary integral is 0 by the given boundary conditions.
- Consequently, $\frac{dE}{dt} = -2\kappa\int_D |\nabla u|^2\,dV \leqslant 0$.

- *(Sec. 2.4)* Heat equation on the line.
  - Let δ_0 be a "point source of heat", of strength 1, located at the origin. Suppose u solves the heat equation with initial data δ_0.
  - Then $v(x,t) = u(\alpha x, \alpha^2 t)$ also solves the heat equation, with initial data a "point source of heat".
  - $\int_{\mathbb{R}} v(x,t)\,dx = \frac{1}{\alpha}\int_{\mathbb{R}} u(x,t)\,dx$. So the initial data for v should be a point source with strength $1/\alpha$.

Lecture 12, Fri 02/10.

- Consequently, $w(x,t) = \alpha u(\alpha x, \alpha^2 t)$ is a solution to the heat equation with initial data δ_0. Consequently $w = u$.
- Consequently, $u(x,t) = \frac{1}{\sqrt{t}}u(\frac{x}{\sqrt{t}}, 1)$. Let $r = x/\sqrt{t}$, and $f(r) = u(r,1)$. Then $u(x,t) = 1/\sqrt{t}f(x/\sqrt{t})$.
- Heat equation reduces to the ODE $f'' + rf' + f = 0$. Solve this:
  - $(f' + rf)' = f'' + rf' + f = 0$. So $f' + rf = c_1$.
  - $e^{r^2/2}$ is an integrating factor: $f = e^{-r^2/2}c_1\int_0^r e^{s^2/2}\,ds + c_2 e^{-r^2/2}$.
  - Since $f \to 0$ at $\pm\infty$, must have $c_1 = 0$.
  - $\int_{-\infty}^{\infty} u(x,1)\,dx = \int_{-\infty}^{\infty} f(r)\,dr = 1 \implies c_2 = \frac{1}{\sqrt{2\pi}}$.
- Thus $u(x,t) = G(x,t) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}$ is the solution of the heat equation on the line, with $\kappa = \frac{1}{2}$ and initial data δ_0 (a point source of strength 1, located at the origin).

Lecture 13, Mon 02/13.

- Translating we see $u(x,t) = G(x-y,t)$ is the solution to the heat equation with initial data δ_y (a point source of strength 1 located at y).
- Approximate f by $\sum \delta_{y_i} f(y_i)(y_{i+1} - y_i)$, and see that $u(x,t) = \int_{-\infty}^{\infty} f(y)G(x-y,t)$ is the desired solution with initial data f.
- Check that this works: $\partial_t - \frac{1}{2}\partial_x^2 u = \int_{-\infty}^{\infty} f(y)(\partial_t - \frac{1}{2}\partial_x^2)G(x-y,t)\,dy = 0$.
- Checking $u(x,0) = f(x)$ is harder (will return to that next week).
- Rescaling time gives $u(x,t) = \int_{-\infty}^{\infty} f(y)G(x-y,2\kappa t)\,dy$ to be the solution to $\partial_t u - \kappa\partial_x^2 u = 0$ with initial data f.
- Formula above shows that for any $t > 0$, u is infinitely differentiable in both x and t, *regardless* of how differentiable f was.
- Infinite speed of propagation. Domain of dependence of the point (x,t) is $(-\infty,\infty)$.

Lecture 14, Wed 02/15.

- Define the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy$.
- Write solutions in terms of the erf.
  - E.g. $f(x) = 1$ for $x \leqslant 0$, and $f(x) = 0$ for $x > 0$.
  - $u(x,t) = \frac{1}{2}(1 - \operatorname{erf}(x/\sqrt{4\kappa t})$ solves the heat equation $\partial_t u - \kappa\partial_x^2 u = 0$ with $u(x,0) = f(x)$ (for all $x \neq 0$).
- Strong maximum principle: If f is continuous and not identically constant, $u(x,t) < \max f$ for any $t > 0$.
  - Proof: $u(x,t) = \int f(y)G(x-y,2\kappa t)\,dy < \int(\max f)G(x-y,2\kappa t)\,dy = \max f$. (Crucially uses the fact that $\int_{-\infty}^{\infty} G(x,t)\,dx = 1$ and $G(x,t) > 0$.)

Lecture 15, Fri 02/17: Midterm.

- In class, closed book. Covers everything up to Lecture 11 *(Sec. 2.3)*.

Lecture 16, Mon 02/20.

- *(Sec. 2.4)* Comparison between the heat and wave equation.
  - Initial data: Heat equation requires $u(x,0)$, wave requires $u(x,0)$ and $\partial_t u(x,0)$.
  - Smoothness of solutions: For $t > 0$, solutions to the heat equation are infinitely differentiable, no matter how many times differentiable (or not differentiable) the initial data is. Solutions to the wave equation are only as differentiable as the initial data.
  - Time reversibility: If u is a solution of the wave equation, then $v(x,t) = u(x, T-t)$ is also a solution (i.e. reversing time, gets back a solution to the wave equation). For the heat equation, reversing time gets the "backward heat equation" which forces heat to collect at hot points.
- *(Sec. 3.3 & 3.4)* Duhamel's principle
  - ODE version: $\dot{y} - Ay = g(t)$, with $y(0) = x$.
    - Let $S(x,t) = e^{At}x$ be the solution operator of the *homogeneous* equation $\dot{y} = Ay$, with $y(0) = x$.
    - The solution to $\dot{y} - Ay = g(t)$ is $S(x,t) + \int_0^t S(g(s), t-s)\,ds$.

Lecture 17, Wed 02/22.

- Heat equation: $\partial_t u - \kappa\partial_x^2 u = g$, $u(x,0) = f(x)$.
  - The solution operator $S(\cdot,t)$ takes *functions* as the first argument, and outputs *functions*.
  - Let $S(f,t)(x) = \int_{-\infty}^{\infty} f(y)G(x-y,2\kappa t)\,dy$.
  - Then $u(x,t) = S(f,t)(x) + \int_0^t S(g(\cdot,s),t-s)\,ds$ is the desired solution.
  - Explicitly $u(x,t) = \int_{-\infty}^{\infty} f(y)G(x-y,2\kappa t)\,dy + \int_0^t\int_{-\infty}^{\infty} g(y,s)G(x-y,t-s)\,dy\,ds$.
- 2nd Order ODE: $\ddot{y} - Ay = h(t)$, with $y(0) = a$ and $\dot{y}(0) = b$.
  - Let $S(x,t)$ solve $\partial_t^2 S - AS = 0$, with $S(x,0) = 0$ and $\partial_t S(x,0) = x$.
  - Then $y(t) = \partial_t S(a,t) + S(b,t) + \int_0^t S(g(s),t-s)\,ds$ is the desired solution.

Lecture 18, Fri 02/24.

- Wave equation: $\partial_t^2 u - c^2 \partial_x^2 u = h$, $u(x,0) = f(x)$, $\partial_t u(x,0) = g(x)$.
  * Let $S(\psi,t)(x) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy$.
  * Then $u(x,t) = \partial_t S(\varphi,t)(x) + S(\psi,t)(x) + \int_0^t S(g(\cdot,s), t-s)\,ds$.
  * Explicitly, $u(x,t) = \frac{1}{2}[\varphi(x+ct)+\varphi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi + \frac{1}{2c}\iint_\Delta h$, where Δ is the domain of dependence of (x,t).

- *(Sec. 3.5)* Continuity of the heat equation at $t=0$.
  - G is an "approximate identity". Namely, G has the following properties:
    * $G(x,t) \geqslant 0$
    * $\int_{-\infty}^{\infty} G(x,t)\,dx = 1$
    * For any $\delta > 0$, $\lim_{t\to 0^+} \int_{|x|\geqslant\delta} G(x,t)\,dx \to 0$.

Lecture 19, Mon 02/27.

- If f is bounded, and continuous at x, then $\lim_{t\to 0} u(x,t) = f(x)$.
  * Pick any $\varepsilon > 0$. Then $\exists \delta > 0$ such that whenever $|z| < \delta$, we have $|f(z) - f(x)| < \varepsilon$.
  * $|u(x,t) - f(x)| = |\int_{-\infty}^{\infty} G(y,t)\,[f(x-y) - f(x)]\,dy| = \int_{|y|<\delta}(\cdot) + \int_{|y|\geqslant\delta}(\cdot)$.
  * First term: $\int_{|y|<\delta}(\cdot) \leqslant \varepsilon \int_{|y|<\delta} G(y,t)\,dy \leqslant \varepsilon$
  * Second term: $\int_{|y|<\delta}(\cdot) \leqslant 2(\max f)\int_{|y|<\delta} G(y,t)\,dy \to 0$ as $t \to 0$.
  * So for t small, can make $|u(x,t) - f(x)| < 2\varepsilon$. QED.

- *(Sec. 4.1)* Separation of variables.
  - Wave Equation, Dirichlet B.C.
    * $u(x,t) = X(x)T(t)$. Then $\frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda$.
    * Hence $X(x) = \alpha\sin(\sqrt{\lambda}x) + \beta\cos(\sqrt{\lambda}x)$.
    * $X(0) = X(L) = 0 \implies \lambda = \frac{n^2\pi^2}{L^2}$.
    * Solve for T: $T(t) = A\sin(\frac{n\pi}{L}ct) + B\cos(\frac{n\pi}{L}ct)$.

Lecture 20, Wed 02/29.

  * $X_n = \sin(\frac{n\pi}{L}x)$, $T_n = A_n\cos(\frac{n\pi}{L}ct) + B_n\sin(\frac{n\pi}{L}ct)$, $u_n(x,t) = X_n(x)T_n(t)$.
  * Frequency of note heard is $\frac{nc}{2L}$. All frequencies heard are multiples of $c/2L$!
  * $u(x,t) = \sum_1^\infty X_n T_n$ is a solution to the wave equation, with $u(x,0) = \sum_1^\infty A_n \sin(\frac{n\pi}{L}x)$, and $\partial_t u(x,0) = B_n \frac{n\pi c}{L}\sin(\frac{n\pi}{L}x)$.
  - Heat Equation, Dirichet B.C.
    * If $u(x,t) = X(x)T(t)$ is a separated solution, then $\frac{X''}{X} = \frac{T''}{c^2T} = -\lambda$.
    * As before, $\lambda = \frac{n^2\pi^2}{L^2}$, $X(x) = \sin(\frac{n\pi}{L}x)$.
    * Solve for T: $T(t) = A_n e^{-\frac{n^2\pi^2}{L^2}\kappa t}$
    * $u(x,t) = \sum_1^\infty X_n(x)T_n(t) = \sum_1^\infty A_n e^{-\frac{n^2\pi^2}{L^2}\kappa t}\sin(\frac{n\pi}{L}x)$, solves the heat equation with Dirichlet B.C. and I.D. $u(x,0) = \sum A_n\sin(\frac{n\pi}{L}x)$.
  - Positivity of λ:
    * If $X'' = -\lambda X$, with $X(0) = X(L) = 0$, then $\lambda = \frac{\int_0^L (X')^2}{\int_0^L X^2} > 0$.

Lecture 21, Fri 03/02.

  - Heat equation, Neumann B.C.:
    * If $u(x,t) = X(x)T(t)$ is a separated solution, then $\frac{X''}{X} = \frac{T''}{c^2T} = -\lambda$.
    * As before, $X(x) = \alpha\cos(\sqrt{\lambda}x) + \beta\sin(\sqrt{\lambda}x)$.
    * B.C. $\implies X'(0) = X'(L) = 0 \implies \lambda = 0$ with $X(x) = \alpha$, or $\lambda = \frac{n^2\pi^2}{L^2}$ with $X(x) = \alpha\cos(\frac{n\pi}{L}x)$.
    * Solve for T: $T(t) = A_n e^{-\frac{n^2\pi^2}{L^2}\kappa t}$
    * $u(x,t) = \sum_0^\infty X_n(x)T_n(t) = \frac{A_0}{2} + \sum_1^\infty A_n e^{-\frac{n^2\pi^2}{L^2}\kappa t}\cos(\frac{n\pi}{L}x)$, solves the heat equation with Neumann B.C. and I.D. $u(x,0) = \frac{A_0}{2} + \sum_1^\infty A_n\cos(\frac{n\pi}{L}x)$.
  - Wave Equation, Neumann B.C.: Similar.

- *(Sec. 5.1)* Fourier Series
  - Goal: Find the coefficients in the Fourier Sine and Cosine series.
  - *Lemma (Symmetry):* $f(0) = g(0) = 0$ and $f(L) = g(L) = 0$ implies $\langle f'', g\rangle = \langle f, g''\rangle$. Here $\langle h_1, h_2\rangle = \int_0^L h_1(x)h_2(x)\,dx$.
    * Proof: Integrate by parts twice. *(Sec. 5.3)*

Lecture 22, Mon 03/05.

  - *Lemma (Orthogonality):* If $X_n(0) = X_n(L) = 0$, $X_n'' = -\lambda_n X_n$, then $\langle X_m, X_n\rangle = 0$ whenever $\lambda_m \leqslant \lambda_n$.
    * Proof: $-\lambda_m\langle X_m, X_n\rangle = \langle X_m'', X_n\rangle = \langle X_m, X_n''\rangle = -\lambda_n\langle X_m, X_n\rangle$.
    * Consequently $(\lambda_m - \lambda_n)\langle X_m, X_n\rangle = 0$. *(Sec. 5.3)*

- Fourier Sine series.
  - Let $X_n = \sin(\frac{n\pi}{L}x)$, $\lambda_n = \frac{n^2\pi^2}{L^2}$.
  - If $f = \sum_1^\infty B_n X_n$, then $B_n = \dfrac{\langle f, X_n\rangle}{\langle X_n, X_n\rangle} = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin(\frac{n\pi}{L}x)\,dx$.
    * Proof: $\langle f, X_m\rangle = \sum_{n=1}^\infty B_n\langle X_n, X_m\rangle = B_m\langle X_n, X_n\rangle$, by the lemma.

- Fourier Cosine Series.
  - Let $X_n = \cos(\frac{n\pi}{L}x)$, and $X_0 = 1$.
  - Verify the Symmetry and Orthogonality lemmas for functions with Neumann Boundary conditions (i.e. $X'(0) = X'(L) = 0$).
  - If $f(x) = \frac{A_0}{2}X_0 + \sum_1^\infty A_n X_n$, then $\frac{A_0}{2} = \frac{\langle f, X_0\rangle}{\langle X_0, X_0\rangle}$, and $A_n = \frac{\langle f, X_n\rangle}{\langle X_n, X_n\rangle}$.
  - Consequently, $A_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos(\frac{n\pi}{L}x)\,dx$, for $n = 0, 1, 2, \ldots$.

Lecture 23, Wed 03/07: Digression – The Black Scholes formula.

- Application of the Heat Equation to Option Pricing (by Kasper Larsen).

Lecture 24, Mon 03/19.

- *(Sec. 5.2)* Full Fourier series.
  - Periodic boundary conditions: $f(x+2L) = f(x)$ for all x.
  - Write $f(x) = \frac{A_0}{2} + \sum_1^\infty A_n \cos(\frac{n\pi}{L}x) + B_n \sin(\frac{n\pi}{L}x)$.
  - The symmetry and orthogonality lemmas give $\int_{-L}^{L} \sin(\frac{m\pi}{L}x)\cos(\frac{n\pi}{L}x)\,dx = \int_{-L}^{L} \sin(\frac{m\pi}{L}x)\sin(\frac{n\pi}{L}x)\,dx = \int_{-L}^{L} \cos(\frac{m\pi}{L}x)\cos(\frac{n\pi}{L}x)\,dx = 0$.
  - Explicitly compute $\int_{-L}^{L} \sin(\frac{n\pi}{L}x)\cos(\frac{n\pi}{L}x)\,dx = 0$.
  - Get $A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)\,dx$ and $B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)\,dx$.
- *(Sec. 5.2)* Complex Fourier series.
  - Suppose f is $2L$ periodic and complex valued.
  - Let $e_n(x) = \exp(i\frac{n\pi}{L}x)$, and want $f(x) = \sum_{-\infty}^{\infty} c_n e_n(x)$.
  - Define $\langle g, h\rangle = \int_{-L}^{L} g(x)\overline{h(x)}\,dx$.
  - Directly check $\langle e_n, e_m\rangle = 0$ if $n \neq m$. (This also follows from the symmetry, orthogonality lemmas).

Lecture 25, Wed 03/21.

  - Conclude $c_n = \langle f, e_n\rangle / \langle e_n, e_n\rangle = \frac{1}{2L}\int_{-L}^{L} f(x)\exp(-i\frac{n\pi}{L}x)\,dx$.
- *(Sec. 5.3/5.4)* Convergence of Fourier series
  - Pointwise, uniform and L^2 convergence.
  - Uniform convergence implies pointwise convergence, but not conversely.
  - Uniform convergence on a *finite interval* implies L^2 convergence, but not conversely.
  - Pointwise need not imply L^2 convergence; L^2 convergence need not imply pointwise convergence.
  - $f_n(x) = 1$ if $x \in (n, n+1)$ and 0 otherwise. Then $(f_n) \to 0$ pointwise, but not uniformly or in L^2.

Lecture 26, Fri 03/23: Midterm.

- In class, closed book. Covers everything from Lecture 11 to Lecture 22.

Lecture 27, Mon 03/26.

  - Let $f_n(x) = 1/2^k$ if $2^k \leqslant n < 2^{k+1}$, $x \in [\frac{n}{2^k}, \frac{n+1}{2^k})$, and $f_n(x) = 0$ otherwise. Then $(f_n) \to 0$ in $L^2[0,1]$, but not pointwise or uniformly.
  - Proof that uniform convergence on a finite interval implies L^2 convergence.
  - Pointwise, uniform and L^2 convergence of series of functions.
    * These are defined as the respective convergence of the partial sums.
  - Bessel's inequality: $\sum_1^\infty A_n^2 \|X_n\|^2 \leqslant \|f\|^2$. (Here $\|f\|^2 = \int_0^L |f(x)|^2\,dx$, and X_n are an orthogonal system, and $A_n = \frac{\langle f, X_n\rangle}{\langle X_n, X_n\rangle}$.)
    * Pythagoras theorem: If $\langle f, g\rangle = 0$, then $\|f+g\|^2 = \|f\|^2 + \|g\|^2$.
    * Consequently, $\|S_N f\| = \sum_1^N A_n^2 \|X_n\|^2$. (Here $S_N f = \sum_1^N A_n X_n$)

Lecture 28, Wed 03/28.

    * $\langle f - S_N f, S_N f\rangle = 0$ (since $\langle f - S_N f, X_n\rangle = 0$ for all $n \leqslant N$).
    * Since $f = (f - S_N f) + S_N f$, the above three bullets imply Bessel's inequality.
  - Amongst all functions of the form $P_N \overset{\text{def}}{=} \sum_1^N b_n X_N$, $S_N f$ is the one that best approximates f in the L^2 norm
    * Proof: Let $E_N f = f - S_N f$. Then $\langle P_N, E_N f\rangle = 0$.
    * Hence $f - P_N = E_N f + (S_N f - P_N)$. By above $\langle E_N f, S_N f - P_N\rangle = 0$.
    * By the Pythagoras theorem, $\|f - P_N\|^2 \geqslant \|f - S_N\|^2$. QED.
  - Proposition: If P_N is any sequence of functions of the form $P_N = \sum_1^N b_{n,N} X_n$ such that $P_N \to f$ in L^2, then $S_N f \to f$ in L^2.
    * Proof: $\|E_N f\|^2 \leqslant \|f - P_N\|^2$, and the RHS converges to 0. QED.
  - Proposition: The series $\sum_1^\infty A_n X_n$ converges to f in L^2 if and only if $\|f\|^2 = \sum_1^\infty A_n^2 \|X_n\|^2$ (Parseval's identity.)
    * Proof (reverse): Say $\|f\|^2 = \sum_1^\infty A_n^2 \|X_n\|^2$.
    * Then $\|E_N f\|^2 = \|f\|^2 - \|S_N f\|^2 = \|f\|^2 - \sum_1^N A_n^2 \|X_n\|^2 \to 0$. QED.

Lecture 29, Fri 03/30.

  - Convergence of Cesàro sums.
    * Let $S_N f(x) = \sum_{-N}^{N} c_n e^{i\frac{n\pi}{L}x}$, where $c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{n\pi}{L}x}$.
    * Show $S_N f(x) = \int_{-L}^{L} f(y) D_N(x-y)\,dy$, where $D_N(z) = \frac{1}{2L}\sum_{-N}^{N} e^{i\frac{n\pi}{L}z}$.
    * Compute $D_N(z) = \dfrac{\sin\bigl((N+\frac{1}{2})\frac{\pi}{L}x\bigr)}{2L\sin\bigl(\frac{1}{2}\frac{\pi}{L}x\bigr)}$
    * Define $\sigma_N f = \frac{1}{N}\sum_{n=0}^{N-1} S_N f$.
    * Show $\sigma_N f(x) = \int_{-L}^{L} f(y) K_N(x-y)\,dy$, where $K_N(z) = \frac{1}{N}\sum_0^{N-1} D_N(z)$.
    * Can compute $K_N(z) = \dfrac{\sin\bigl(\frac{N}{2}\frac{\pi}{L}x\bigr)^2}{2NL\sin\bigl(\frac{1}{2}\frac{\pi}{L}x\bigr)^2}$ [will be on Homework]
    * Can check $\displaystyle\int_{-L}^{L} K_N(z)\,dz = 1$, $K_N \geqslant 0$, and $\displaystyle\lim_{N\to\infty}\int_{\substack{|z|>\varepsilon\\ z\in[-L,L]}} K_N(z)\,dz = 0$, for any $\varepsilon > 0$.
    * Hence if f is continuous at x, then $\lim_{N\to\infty} \sigma_N f(x) = f(x)$ (on HW).
    * Further, if f is continuous on $[-L, L]$ then $(\sigma_N f) \to f$ *uniformly* on $[-L, L]$.
    * Consequently, $(\sigma_N f) \to f$ on $L^2[-L, L]$, and from last time this implies $(S_N f) \to f$ in $L^2[-L, L]$.

Lecture 30, Mon 04/02.

- General convergence theorems.
  * If $\int_0^L f(x)^2\,dx < \infty$, then the Fourier Sine/Cosine series converges to f in L^2.
  * At any point $x \in (0, L)$ where f is continuous, then then Cesàro sums converge but the partial sums need not.
  * Consequently, if f is continuous on $[0, L]$ and satisfies the boundary conditions, then then then Cesàro sums converge pointwise, converge but the partial sums need not.
  * If f is differentiable at $x \in (0, L)$ then the Fourier Sine/Cosine series converges to f at x.
  * Consequently, if f satisfies the boundary conditions, and is continuous and piecewise differentiable then the Fourier Sine/Cosine series converge pointwise to f.
  * If further $\int_0^L f'(x)^2\,dx < \infty$, then the Fourier Sine/Cosine series converge *uniformly.*
  * Analogous results hold for the full / complex Fourier series.
- Fourier coefficients of f'.
  * Suppose f is periodic, differentiable and $\int_{-L}^L f'(x)^2\,dx < \infty$.
  * Let c_n be the (complex) Fourier coefficients of f, and d_n those of f'. Then $d_n = i\frac{n\pi}{L}c_n$.
  * Proof: Differentiating term by term is NOT JUSTIFIABLE! However, using the formula for d_n and integrating by parts gives the desired relation.
- Sobolev embedding: If $\sum |n^s c_n|^2 < \infty$ for $s > \frac{1}{2}$, then f is continuous! (If $s > 3/2$, then f is differentiable, and f' is continuous.)
  * Proof: Try it yourself, if you konw the Cauchy-Schwartz inequality and the Weirstrauss M test.

Lecture 31, Wed 04/04.

- *(Sec. 6.1)* Laplace and poisson equation.
  - Laplace equation in upper half plane: $\partial_t^2 u + \partial_x^2 = 0$, for $t > 0$ and $x \in \mathbb{R}$.
    * Differs from the wave equation by only a sign.
    * Laplace equation only reqires the "initial position", OR the "initial velocity". Wave equation requires both.
    * Laplace equation satisfies a maximum principle. Wave does not.
    * Solutions to the Laplace equation are smooth for any $t > 0$. Solutions to the wave equation are only as differentiable as the initial data.
    * Wave equation has finite speed of propagation. The laplace equation does not.
  - Motivation: Steady states of the heat equation.
    * The equilibrium temperature in a conductor satisfies $-\triangle u = f$, where f is the sources / sinks of heat.
    * Dirichlet boundary conditions correspond to holding the temperature at the boundary constant. Neumann boundary conditions correspond to insulating the conductor.
  - Electrodynamics: Electric electric potential.
    * Maxwell's equations reduce to $-\triangle u = \rho$, where u is the electric potential and ρ is proportional to the charge density.
    * Dirichlet boundary conditions correspond to specifying the voltage at the boundary. Neumann boundary conditions correspond to specifying the current.
  - Harmonic functions are functions that satisfy $-\triangle u = 0$.
    * The charge density in a perfect conductor is 0, so the electric potential is harmonic.
    * Equilibrium temperature in a conductor (in the absence of sources / sinks) is also a harmonic function.
  - Uniqueness: Suppose u_1, u_2 are solutions of $-\triangle u = f$ in Ω with $u = g$ on $\partial\Omega$. Then $u_1 = u_2$.
    * Proof: Let $v = u_1 - u_2$. Then $-\triangle v = 0$ in Ω, and $v = 0$ on $\partial\Omega$.
    * By the divergence theorem $-\int_\Omega v\triangle v = -\int_{\partial\Omega} v\frac{\partial v}{\partial \hat{n}} + \int_\Omega |\nabla v|^2$.
    * Consequently $\int_\Omega |\nabla v|^2 = 0$, forcing v to be a constant. Since $v = 0$ on $\partial\Omega$, we must have $v = 0$ identically.

Lecture 32, Fri 04/06.

- Laplacian in polar coordinates.
  - Put $x = r\cos\theta$, $y = r\sin\theta$.
  - Get $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$.
  - Compute $\partial_x r = \cos\theta$, $\partial_y r = \sin\theta$, $\partial_x\theta = -\frac{1}{r}\sin\theta$, and $\partial_y\theta = \frac{1}{r}\cos\theta$.
  - Compute $\partial_x u = \cos\theta\partial_r u - \frac{1}{r}\sin\theta\partial_\theta u$, and $\partial_y u = \sin\theta\partial_r u + \frac{1}{r}\cos\theta\partial_\theta u$
  - Compute $\triangle u = \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u$.
- *(Sec. 6.3)* Laplace equation in a disk.
  - Let D be a disc with center 0 and radius a.
  - Let $-\triangle u = 0$ in D, with $u = f$ on ∂D.
  - Switch to polar coordinates and separate variables.
    * Let $u(r, \theta) = R(r)T(\theta)$.
    * Get $\frac{r^2R'' + rR'}{R} = -\frac{T''}{T} = \lambda$
    * Periodic boundary conditions on T gives $T(\theta) = \cos(n\theta)$, $T(\theta) = \sin(\theta)$, or $T(\theta) = 1$, and $\lambda = n^2$ for $n \in \{0, 1, \dots\}$.

Lecture 33, Mon 04/09.

* Try $R(r) = r^\alpha$ as a solution.
* Get $\alpha(\alpha - 1) + \alpha = n^2$, and hence $\alpha = \pm n$.
* Reject $\alpha = -n$ as $X(r) = r^{-n}$ blows up at $r = 0$.

  - $*$ Thus separated solutions are of the form $r^n\cos(n\theta)$, $r^n\sin(n\theta)$ and constants.
  - $*$ If the full Fourier series of f is $f(\theta) = \frac{A_0}{2} + \sum_1^\infty A_n\cos(n\theta) + B_n\sin(n\theta)$, then $u(r,\theta) = \frac{A_0}{2} + \sum_1^\infty A_n\frac{r^n}{a^n}\cos(n\theta) + B_n\frac{r^n}{a^n}\sin(n\theta)$.

- Harmonic functions in a disk.
  - Use complex notation. $(x,y) = z = re^{i\theta}$.
  - Write $f(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta}$.
  - Then $u(x,y) = \sum_{-\infty}^{\infty} c_n \frac{r^n}{a^n} e^{in\theta}$.
  - Since $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)e^{-in\phi}\,d\phi$, we have $u(r,\theta) = \int_{-\pi}^{\pi} P(r,\theta-\phi)\,f(\theta)\,d\theta$, where $P(r,\theta) = \frac{1}{2\pi}\left[1 + \sum_1^\infty \frac{r^n}{a^n}(e^{in\theta} + e^{-in\theta})\right]$.
  - Compute $P(r,\theta) = \frac{1}{2\pi}\frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta)}$
  - As $r \to a^-$, P behaves like an approximate identity:
    - $*$ $\int_{-\pi}^{\pi} P(r,\theta)\,d\theta = 1$ (Proof: $P(r,\theta) = \frac{1}{2\pi}(1 + \sum 2(\frac{r}{a})^n\cos(n\theta))$.)
    - $*$ $P(r,\theta) > 0$ for $0 \leqslant r < a$. (Proof: $|\cos\theta| \leqslant 1$, so $P(r,\theta) \geqslant \frac{a^2-r^2}{2\pi(a+r)^2} > 0$.)
    - $*$ For any $\varepsilon > 0$, $\lim\limits_{r\to a^-}\int_{|\theta|>\varepsilon} P(r,\theta) = 0$. (Proof: on HW).

Lecture 34, Wed 04/11.

  - $\lim_{r\to a^-} P(r,\theta) = 0$ if $\theta \neq 0$, and ∞ otherwise.
    - $*$ Proof: $P(r,\theta) = \frac{1}{2\pi}\frac{a^2-r^2}{(a-r)^2+2ar(1-\cos\theta)}$.

- Poissons formula:
  - If $-\triangle u = 0$ in some domain D, and a disk with center x_0, and radius a is completely contained inside D, then $u(x) = \frac{a^2-|x-x_0|^2}{2\pi a}\int_{|y-x_0|=a}\frac{u(y)}{|x-y|^2}\,dy$ whenever $|x-x_0| < a$.
- Mean value property: $\triangle u = 0$, then $u(x) = \frac{1}{2\pi r}\oint_{|y-x|=r} u(y)\,dy$. [Note, the RHS is a *line integral* over the circle with center x_0 and radius a.]
  - Proof: use the Poisson formula.
- Strong maximum principle: If $\triangle u = 0$ in D, then u attains it's maximum (and minimum) only on ∂D, unless u is constant.
  - Proof: Use the mean value property.

Lecture 35, Fri 04/13.

- *Weak Maximum principle including convection terms.* Let $Lu = -\triangle u + b\cdot\nabla u$, with b bounded. Suppose $Lu \leqslant 0$ in a (bounded) domain D, and u is continuous up to the boundary of D. Then u attains it's maximum on ∂D (i.e., $\max\limits_D u \leqslant \max\limits_{\partial D} u$).
  - Lemma: Suppose first $Lv < 0$. Then v has no interior maximum in D.
    - $*$ Proof: At an interior maximum x_0, $\nabla u = 0$ and $\triangle u \geqslant 0$, contradicting $Lu < 0$.
    - $*$ Now let $v = u + \varepsilon e^{\lambda x}$, for λ to be chosen later.
    - $*$ Compute $Lv = Lu + \varepsilon\left(-\lambda^2 e^{\lambda x} + \lambda b_1\right)e^{\lambda x}$.
    - $*$ Know $Lu \leqslant 0$. Choosing $\lambda < \max b_1$ will guarantee the second term is stricitly negative.
    - $*$ So $Lv < 0$, and by the Lemma $\max_D v \leqslant \max_{\partial D} v$.
    - $*$ As before, $\max\limits_D u \leqslant \max\limits_D v \leqslant \max\limits_{\partial D} v \leqslant \max\limits_{\partial D} u + \varepsilon e^{\lambda L}$, and send $\varepsilon \to 0$.

Lecture 36, Mon 04/16.

- *(Sec. 7.1)* Greens Identities
  - Greens first identity: $D \subseteq \mathbb{R}^3$ bounded. $\int_D u\triangle v + \int_D \nabla u\cdot\nabla v = \int_{\partial D} u\frac{\partial v}{\partial n}$.
    - $*$ Proof: Divergence theorem applied to $\nabla\cdot(u\nabla v) = u\triangle v + \nabla u\cdot\nabla v$.
  - Dirichlet's principle: The function minimising the energy $E(u) \stackrel{\text{def}}{=} \int_D |\nabla u|^2$ subject to $u = f$ on ∂D, is the harmonic function with boundary values f.
    - $*$ Proof: Let v be any function which is 0 on ∂D. Must have $\frac{d}{d\varepsilon}E(u+\varepsilon v) = 0$ when $\varepsilon = 0$.
    - $*$ Greens identity gives $\int_D v\triangle u = 0$.
    - $*$ Since this is true for any v, must have $\triangle u = 0$.
  - The above shows that if u minimises E, then we must have $\triangle u = 0$. We need to also check the converse: Namely if $\triangle u = 0$, then u minimises E.
    - $*$ Proof: Let $u, v = f$ on ∂D, and suppose $\triangle u = 0$ in D.
    - $*$ Set $w = u - v$; then $v = u - w$ and $w = 0$ on ∂D.
    - $*$ Green's identity implies $\int_D (\nabla u)\cdot(\nabla w) = \int_{\partial D} w\frac{\partial u}{\partial\hat{n}} - \int_D w\triangle u = 0$.
    - $*$ Thus $E(v) = \int_D |\nabla u|^2 + |\nabla w|^2 - 2(\nabla u)\cdot(\nabla w) = E(u) + E(w) \geqslant E(u)$.

Lecture 37, Wed 04/18.

- The Rayleigh quotient: Let $E(u) = (\int_D |\nabla u|^2)/\int_D u^2$. Minimise E, over all functions u which are 0 on ∂D.
  - Let $\varepsilon \in \mathbb{R}$, and v be any function which is 0 on ∂D.
  - Suppose $\lambda = \min E$, and is attained for the function φ.
  - Green's identity shows $\frac{d}{d\varepsilon}(\varphi + \varepsilon v)\big|_{\varepsilon=0} = 0$ iff $\int_D v(-\triangle\varphi - \lambda\varphi) = 0$.
  - Eigenfunctions of the Laplacian are candidates for the minimiser of E.
- Let φ solve $-\triangle\varphi = \lambda\varphi$, with $\varphi = 0$ on ∂D, and $\varphi > 0$ in D.
  - This is called the Principal eigenfunction. It's existence (positivity) is equivalent to the maximum principle!
  - Claim: φ minimises E.
    - $*$ Proof: Let $\varepsilon > 0$, and u be any function with $u = 0$ on ∂D.
    - $*$ Compute $\lambda\int_D u^2\frac{\varphi}{\varphi+\varepsilon} = -\int_D \frac{u^2}{\varphi+\varepsilon}\triangle\varphi \leqslant \int_D |\nabla u|^2$ (Greens identity, and completing the square).
    - $*$ Send $\varepsilon \to 0$. QED.

Lecture 38, Mon 04/23.

- *(Sec. 7.2)* Greens second identity: $\int_{\partial D} u\frac{\partial v}{\partial \hat{n}} - v\frac{\partial u}{\partial \hat{n}} = \int_D u\triangle v - v\triangle u$.
  - Divergence theorem applied to $\nabla \cdot (u\nabla v - v\nabla u) = u\triangle v - v\triangle u$.
- Mean value property: If $\triangle u = 0$ in B_R, a $3D$ ball of radius R and center x_0. Then $u(x_0) = \frac{1}{4\pi R^2}\int_{\partial B_R} u(x)\,dx$ (the RHS is a surface integral).
  - Proof: Without loss assume $x_0 = 0$.
  - Let $v(x) = \frac{1}{|x|}$. Know (from HW) $\triangle v = 0$ for $x \neq 0$.
  - For $\varepsilon > 0$, let $D_\varepsilon = B_R - \bar{B}_\varepsilon = \{x \in \mathbb{R}^3 \mid \varepsilon < |x| < R\}$.
  - Greens identity implies $\int_{D_\varepsilon} u\frac{\partial v}{\partial \hat{n}} - v\frac{\partial u}{\partial \hat{n}} = 0$.
  - $\int_{D_\varepsilon}(\cdot) = \int_{B_R}(\cdot) + \int_{B_\varepsilon}(\cdot)$. (The normal derivative points radially inward on ∂B_ε, and radially outward on ∂B_R.)
  - Compute $\int_{\partial B_R} u\frac{\partial v}{\partial \hat{n}} - v\frac{\partial u}{\partial \hat{n}} = -\frac{1}{R^2}\int_{\partial B_R} u(x)\,dx$.
  - Similarly $\int_{\partial B_\varepsilon} u\frac{\partial v}{\partial \hat{n}} - v\frac{\partial u}{\partial \hat{n}} = \frac{1}{\varepsilon^2}\int_{\partial B_\varepsilon} u(x)\,dx \xrightarrow{\varepsilon\to 0} 4\pi\,u(0)$, since u is continuous at 0.
  - Since $\int_{\partial B_R}(\cdot) = \int_{B_\varepsilon}(\cdot)$, we get $u(0) = \frac{1}{4\pi R^2}\int_{\partial B_R} u(x)\,dx$. QED.

Lecture 39, Wed 04/25.

- Representation formula: If $D \subseteq \mathbb{R}^3$, and $\triangle u = 0$ in D. Then
$$u(x_0) = \int_{\partial D} [u(x)\frac{\partial N}{\partial \hat{n}}(x - x_0) - N(x - x_0)\frac{\partial u}{\partial \hat{n}}(x)]\,dx,$$
where $N(x) = \frac{1}{4\pi|x|}$ is the Newton potential. (Recall $\triangle N = 0$ when $x \neq 0$).
  - Remark: For a $2D$ domain, the same formula is true with $N(x) = \frac{1}{2\pi}\ln|x|$.
  - Proof: Without loss, $x_0 = 0$. Let $D_\varepsilon = D - \{x \in D \mid |x| > \varepsilon\}$, and $B_\varepsilon = \{x \mid |x| < \varepsilon\}$.
  - Greens identity implies $\int_{\partial D_\varepsilon} u\frac{\partial N}{\partial \hat{n}} - N\frac{\partial u}{\partial \hat{n}} = 0$.
  - $\int_{\partial D_\varepsilon}(\cdot) = \int_{\partial D}(\cdot) + \int_{\partial B_\varepsilon}(\cdot)$, where the normal derivative points radially inward on ∂B_ε, and outward to D on ∂D.
  - $\int_{\partial D}(\cdot)$ is the RHS we want.
  - $\int_{\partial B_\varepsilon}(\cdot) \xrightarrow{\varepsilon\to 0} -u(x_0)$, exactly as before. QED
- Corollary: Another proof of the Mean value property.
  - Proof: Put $D = \{x \mid |x - x_0| = R$, in the representation formula.

Lecture 40, Fri 04/27.

- Physical intuition behind the Newton potential.
  - N is the steady temperature obtained from a point *sink* of heat located at 0.
  - Symmetry forces $N(x) = f(|x|)$.
  - Computing heat flux through ∂B_R gives $\int_{\partial B_R}\frac{\partial N}{\partial \hat{n}} = 1$.
  - Consequently $4\pi f'(R)^2 = 1$ for all R, forcing $N(x) = \frac{-1}{4\pi|x|}$, as we had.
- *(Sec. 7.3)* Greens functions.
  - $G(x, x_0)$ is the greens function of a domain $D \subseteq \mathbb{R}^3$ if it has the following properties:
    * For $x \neq x_0$, all second order partials of G (w.r.t. x) are continuous and $\triangle G(x, x_0) = 0$.
    * $G(x, x_0) = 0$ on ∂D
    * $H(x) = G(x, x_0) - N(x - x_0)$, for $x \neq x_0$ extends to a continuous harmonic function in D.
  - If $\triangle u = 0$ in D and $u = f$ on ∂D, then $u(x_0) = \int_{\partial D} f(x)\frac{\partial G}{\partial \hat{n}}(x, x_0)$.
    * Proof: Know $u(x_0) = \int_{\partial D} u(x)\frac{\partial N}{\partial \hat{n}}(x - x_0) - N(x - x_0)\frac{\partial u}{\partial \hat{n}}$.
    * By greens identity, $\int_{\partial D} u\frac{\partial H}{\partial \hat{n}} - H\frac{\partial u}{\partial \hat{n}} = 0$.
    * Since $G(x, x_0) = H(x) + N(x - x_0)$, adding the above two identites (and using $G(x, x_0) = 0$ for $x \in \partial D$) finishes the proof.
- Symmetry of Greens functions: $G(a, b) = G(b, a)$.
  - Proof: Put $u(x) = G(x, a)$ and $v(x) = G(x, b)$.
  - Let $D_\varepsilon = \{x \in D \mid |x - a| > \varepsilon \;\&\; |x - b| > \varepsilon\}$.
  - Greens identity: $\int_{\partial D_\varepsilon} u\frac{\partial v}{\partial \hat{n}} - v\frac{\partial u}{\partial \hat{n}} = 0$.
  - $\int_{\partial D_\varepsilon} = \int_{\partial D} + \int_{\partial B(a,\varepsilon)} + \int_{\partial B(b,\varepsilon)}$, with the usual convention about normals.
  - Since $u = v = 0$ on ∂D, $\int_D(\cdot) = 0$.
  - Claim: $\int_{\partial B(a,\varepsilon)} u\frac{\partial v}{\partial \hat{n}} - v\frac{\partial u}{\partial \hat{n}} = v(a)$, and $\int_{\partial B(b,\varepsilon)} u\frac{\partial v}{\partial \hat{n}} - v\frac{\partial u}{\partial \hat{n}} = -u(b)$.
  - Claim finishes the proof, since $v(a) = u(b) \iff G(a, b) = G(b, a)$.

Lecture 41, Mon 04/30.

  - Proof of claim.
    * In $B(a, \varepsilon)$, v is harmonic and $u(x) = H(x) + N(x - a)$ for some harmonic function H.
    * Greens identity implies $\int_{\partial B(a,\varepsilon)} H\frac{\partial v}{\partial \hat{n}} - v\frac{\partial H}{\partial \hat{n}} = 0$.
    * Representation formula implies $\int_{\partial B(a,\varepsilon)} N(x - a)\frac{\partial v}{\partial \hat{n}} - v\frac{\partial N}{\partial \hat{n}}(x - a) = v(a)$. (The sign is reversed, since the normal vector is inward pointing).
    * Adding gives $\int_{\partial B(a,\varepsilon)} u\frac{\partial v}{\partial \hat{n}} - v\frac{\partial u}{\partial \hat{n}} = v(a)$. QED.
  - Physical interpretation: Steady temperature at a caused by a point sink at b is the same as a steady temperature at b caused by a point sink at a.

Lecture 42, Wed 05/02.

- Greens function in Half-Space.
  - $D = \{(x, y, z) \mid z > 0\}$. $G(x, x_0) = N(x - x_0) - N(x - x_0^*)$, where x_0* is the image of the point x_0 reflected about the x-y plane.

- Consequently the solution to $-\triangle u = 0$ in D with $u = f$ on ∂D is given by $u(x) = \displaystyle\int_{\partial D} f(y) G(x,y)\,dy = \frac{x_3}{2\pi} \iint \frac{f(y_1,y_2,0)\,dy_1\,dy_2}{[(x_1-y_1)^2+(x_2-y_2)^2+x_3^2]^{3/2}}$.

- Greens function in a shpere
  - $D = \{x \in \mathbb{R}^3 \mid |x| < a\}$.
  - Let $x_0 \in D$, and $x_0^* = \frac{a}{|x_0|^2} x_0$. Then $G(x,x_0) = N(x-x_0) + N(\frac{|x_0|}{a}(x-x_0^*))$.
  - Basic congruent triangles argument shows $G(x,x_0) = 0$ when $x \in \partial D$.
  - Compute $\dfrac{\partial G}{\partial \hat{n}} = \dfrac{a^2 - |x_0|^2}{4\pi a |x-x_0|^3}$.
  - 3D Poisson formula. If $\triangle u = 0$ in D and $u = f$ on ∂D, then $u(x_0) = \int_{x\in\partial D} \frac{a^2-|x_0|^2}{4\pi a|x-x_0|^3} f(x)\,dx$. (Surface integral over the sphere of radius a)