Questions from previous exams.

This is a collection of questions I've used in previous finals, which are relevant to the material tested in Midterm 2. I don't have solutions written up for them, but can solve them for you if you come by my office hours. The amount of time you have per question is roughly 1.2 times the number of points shown in the margin.

[5] 1. (a) Let u be a solution of the PDE $u_t + u_x - \frac{1}{2}u_{xx} = 0$ for $x \in \mathbb{R}$, $t > 0$ with initial data $u(x, 0) = f(x)$. Find a formula expressing u in terms of f. [HINT: Put $v(x, t) = u(x + t, t)$, and find a PDE satisfied by v.]

[5] (b) Use Duhamel's formula and the previous part to find the solution of the PDE $u_t + u_x - \frac{1}{2}u_{xx} = g$ for $x \in \mathbb{R}$, $t > 0$ with initial data $u(x, 0) = f(x)$. [Here g is some given function of x and t.]

[10] 2. Solve the PDE $\partial_t u - \frac{1}{2}\partial_{xx}u = -u$ when $x \in \mathbb{R}$, $t > 0$ and $u(x, 0) = f(x)$. [Your answer should express $u(x, t)$ in terms of f, x, and t. It can involve nasty integrals. It might help to consider the function $v(x, t) = e^{\pm t}u(x, t)$.]

[10] 3. (a) Suppose u solves the PDE $u_{tt} - u_{xx} = 0$ in the rectangle $x \in (0, \pi)$, $t \in (0, 1)$ with $u = 0$ on the sides, bottom and top of this rectangle. Show that u is the constant function 0. [HINT: Use separation of variables. You can assume that u is a continuous function.]

[5] (b) If we change $t \in (0, 1)$ to $t \in (0, \pi)$ above, then there are *infinitely many* non-zero solutions of the above PDE. Find one of them. [Namely, find a function u such that $u_{tt} - u_{xx} = 0$ in the rectangle $x \in (0, \pi)$, $t \in (0, \pi)$ with $u = 0$ on the sides, bottom and top of this rectangle, but u is not identically the 0 function.]

(c) *(Not on a previous exam.)* Provide an intuitive explanation (without any separation of variables expansions) why you have a unique solutions in the first case, and non-uniqueness in the second.

[5] 4. (a) Suppose f and g are two odd functions of x, and u solves the PDE $u_{tt} - c^2u_{xx} = 0$ with initial data $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$. Show that $u(0, t) = 0$ for all $t > 0$.

[10] (b) Suppose u solves the wave equation $u_{tt} - u_{xx} = 0$ on the region $x, t > 0$, with boundary condition $u(0, t) = 0$ and initial data $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$. What portion of the initial data influences the value of $u(1, 2)$. [If you don't understand what this means, I will instead accept an exact formula for $u(1, 2)$ involving only f and g. Also note that in class I have *not* derived a formula for solutions to the wave equation with these boundary conditions! Thus if you know it, you can only use it if you derive it. If you don't know it (which is as it should be), then to derive the formula remember D'Alembert (which you can blindly use without proof), what I did for the heat equation with similar boundary conditions, and look at the previous subpart.]

[5] 5. (a) Find all *separated* solutions to the PDE $v_t - v_{xx} + v = 0$ with boundary conditions $v(0, t) = 0 = v(L, t)$. Namely if $v_n(x, t) = X_n(x)T_n(t)$ solves the given PDE with boundary conditions, then find the functions X_n, and T_n.

[5] (b) Let $f(x) = \sum_1^\infty A_nX_n(x)$, where X_n is from the previous subpart. Find the solution of the PDE $v_t - v_{xx} + v = 0$ with boundary conditions $v(0, t) = 0 = v(L, t)$ and initial data $v(x, 0) = f(x)$. [HINT: You may assume for this question that that infinite sums can be differentiated as you please. Your answer should express $v(x, t)$ as an infinite sum involving A_n, x, t.]

5 (c) Suppose $L = \ln 2$, and w solves $-w_{xx} + w = 0$ with $w(0) = w(L) = -1$. Find w.

The next two parts require material I haven't covered in class yet. The point of this question was to show the last part, which constructs a counter example to a variant of the maximum principle, so I include them here for completeness. Don't try them unless you know the material.

10 (d) Suppose now that u satisfies the same PDE $u_t - u_{xx} + u = 0$, but with boundary conditions $u(0,t) = -1 = u(L,t)$, and initial data $u(x,0) = -1$. Show that

$$\int_0^L |u(x,t) - w(x)|^2 \, dx \leqslant e^{-2(1+\frac{\pi^2}{L^2})t} \int_0^L |1 + w(x)|^2 \, dx.$$

[HINT: Set $v = u - w$. What PDE, and boundary conditions does v satisfy? Part (b) helps.]

5 (e) Suppose now that

$$\lim_{t \to \infty} \max_{x \in [0,L]} |u(x,t) - w(x)| = 0$$

(extra 5 points if you can either prove this, or solve this subpart without this assumption). Show that for T large enough, u does not attain it's maximum on the sides or bottom of the rectangle $[0,L] \times [0,T]$. [HINT: Draw w.]