Supplement: Method of characteristics.

Consider a PDE of the form

$$a\partial_x u + b\partial_y u = f, \tag{1}$$

where a, b and f are functions of the *spatial* variables x and y. Our aim is to outline a method to find the general solution to this PDE.

The main idea is that we can reduce this to an ODE along 'characteristic curves'. Suppose now x and y are in turn functions of a (dummy) variable t, and satisfy the ODE

$$\frac{dx}{dt} = a(x,y) \qquad \frac{dy}{dt} = b(x,y). \tag{2}$$

Solutions to this ODE are called characteristic curves. Recall, if $f = 0$, then the function u must be constant along such curves. If, however, f is not 0, then u need not be constant along such curves. But, equation (1) reduces to an ODE along such curves. Solving this ODE should tell you how u is changing along this curve. This, in principle, it should be possible to find u explicitly.

Let's try it: Applying the chain rule we see

$$\frac{du}{dt} = \partial_x u \frac{dx}{dt} + \partial_y u \frac{dy}{dt} = a\partial_x u + b\partial_y u.$$

Now using equation, we have the ODE

$$\frac{du}{dt} = f. \tag{3}$$

Now assuming a, b, f are 'nice enough' functions, one can eliminate the dummy variable t from (2) and (3). This gives

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f}.$$

To find the general solution, first solve the ODE

$$\frac{dx}{a} = \frac{dy}{b},$$

and express the solution in the form

$$F(x,y) = c_1, \tag{4}$$

where you find F explicitly, and c_1 is the undetermined constant.

Next solve the ODE

$$\frac{dx}{a} = \frac{du}{f}. \tag{5}$$

To do this, you should first eliminate y completely from the equation. Remember, this equation is solved along a characteristic! So you already have a relation between x and y (equation (4)), so you can eliminate y from this equation and write it in terms of just u, x and the constant c_1. (You could equivalently use the ODE $\frac{dy}{b} = \frac{du}{f}$, and eliminate x from the equation.)

Once you solve the ODE for u, express your answer in the form

$$u = G(x, c_1, c_2), \tag{6}$$

where you find the function G explicitly, and c_2 is the undetermined constant obtained from solving this ODE.

Now the key point is that c_2 must be a (general) function of c_1. This is because you're solving the ODE (5) *along characteristics.* You have to solve it on every single characteristic to get the value of u, and each time you solve it you get the undetermined constant c_2. Now c_2 can vary between characteristics, however *must* be the same on any particular characteristic! Of course, which characteristic you're on is determined by (4) – namely, by c_1. Thus c_2 must be some, general, function of c_1.

Letting g be some general function, substituting $c_2 = g(c_1)$ and $c_1 = F(x,y)$ into (6) gives

$$u = G(x, F(x,y), g(F(x,y)))$$

as the general solution.

Finally, we remark that this method will still work if f is a function of x, y and u. It will also work if we have more than two variables, and some such questions are on your homework. Here's one example to get you started.

Example 1. Find the general solution of $y\partial_x u + x\partial_y u = y(x^2 + y^2)$

Solution. First find the characteristics: Solving the ODE $\frac{dx}{y} = \frac{dy}{x}$ shows the characteristics are described by $x^2 - y^2 = c_1$.

Next, let's solve $\frac{du}{y(x^2+y^2)} = \frac{dx}{y}$. Using our equation for characteristics, we eliminate y. This gives the ODE $\frac{du}{2x^2 - c_1} = dx$. Solving this gives

$$u = \frac{2}{3}x^3 - c_1 x + c_2,$$

for some constant c_2. Writing $c_1 = x^2 - y^2$, and $c_2 = g(c_1)$ for a general function g, we see the general solution has the form

$$u = \frac{2}{3}x^3 - (x^2 - y^2)x + g(x^2 - y^2). \qquad \square$$

Further reading

If you'd like a deeper treatment of the method of characteristics (more rigorous, and deeper) I recommend reading Chapter 1 of Fritz John's book. It's accessible to most undergraduates, however is beyond the scope of this course.