Math 131P Midterm

Tue, Oct. 21, 2008

Time: 75 mins
Total: 60 points

This is a closed book, closed laptop test. Calculators, computational aids, cell phones, pagers, etc. are strictly forbidden. Good luck ⌣̈

[10] 1. Find the general solution to the PDE $y\partial_x u - x\partial_y u = y(x^2 + y^2)$.

[10] 2. Suppose u satisfies the PDE $u_t + u_{xx} = 0$ in the rectangle $R = [0, L] \times [0, T]$ and is continuous up to the boundary of R. At what points can u attain a maximum? Justify. [You may assume that at a maximum $u_{xx} \neq 0$.]

[6] 3. (a) Let $G(x, y, t) = \frac{1}{2\pi t} e^{-\frac{x^2+y^2}{2t}}$. Show that G satisfies the (two dimensional) heat equation

$$\partial_t G - \frac{1}{2} \triangle G = 0.$$

What can you say about the initial data?

[4] (b) Guess a formula for $G(x_1, \dots, x_n, t)$ so that G satisfies

$$\partial_t G - \frac{1}{2} \sum_{i=1}^{n} \partial_{x_i}^2 G = 0,$$

with initial data similar to that in the previous subpart. [You don't have to verify that G indeed satisfies this equation. Just an educated guess is enough.]

[10] 4. Let $u_{tt} - c^2 u_{xx} = 0$, for $x \in \mathbb{R}$, $t > 0$ with initial data $u(x, 0) = \varphi(x)$, $u_t(x, 0) = \psi(x)$. Find functions φ and ψ such that

$$u(x, 10) = \begin{cases} 1 & \text{when } x \geqslant 0 \\ 0 & \text{when } x < 0 \end{cases} \quad \text{and} \quad u_t(x, 10) = 0.$$

[10] 5. Suppose u satisfies the wave equation $u_{tt} - c^2 u_{xx} = 0$ for $x \in (a, b)$, $t > 0$ with initial data $u(x, 0) = \varphi(x)$, $u_t(x, 0) = \psi(x)$ and Neumann boundary conditions $u_x(a, t) = 0 = u_x(b, t)$. Compute $\int_a^b u(x, t)\, dx$ in terms of t, φ and ψ.

[10] 6. Suppose u satisfies the dissipative heat equation $u_t - \kappa u_{xx} = -\alpha u$ for $x \in \mathbb{R}$, $t > 0$ with initial data $u(x, 0) = f(x)$. Show that $u(x, t) \leqslant e^{-\alpha t} \max(f)$. [Hint: Cleverly define a function v in terms of u so that v satisfies $v_t - \kappa v_{xx} = 0$.]