Math 131 Midterm

Tue, Jan 29, 2008

Time: 75 mins
Total: 40 points

This is a closed book test. Please don't use cell phones or pagers. Good luck ☺

5 1. (a) Find the general solution to the PDE $\partial_x u + 2x\partial_y u = 0$.

5 (b) Sketch the largest region in $\mathbb{R}^2$ on which you can uniquely determine the solution to the above PDE, subject to the (auxiliary) condition $u(x, 1) = \sin(x^3)$. [Just a sketch of the region, and an explanation is enough. No need to compute the solution.]

10 2. Let $u(x, y, t)$ be the population of a virus at the point $(x, y) \in \mathbb{R}^2$ and time t. Suppose the virus population changes as follows:

(i) Due to overcrowding, the virus migrates from regions of high population to regions of low population at a rate proportional to the gradient. (More precisely, the rate of migration in a particular direction $\vec{v}$ equals $\kappa(\nabla u) \cdot \vec{v}$, where $\kappa > 0$ is some constant.)

(ii) The rate at which the virus population grows (due to reproduction and death) equals $u(1 - u)$.

Find a PDE satisfied by the function u.

3. Suppose u satisfies the PDE $u_t - u_{xx} + u = 0$, when $x \in (0, 1)$ and $t > 0$, with boundary conditions $u(0, t) = 0 = u(1, t)$ for $t \geqslant 0$, and with initial data $u(x, 0) = f(x)$ for $x \in [0, 1]$.

5 (a) Let $v(x, t) = e^t u(x, t)$. Find a PDE satisfied by v. Also find it's boundary conditions, and initial data.

5 (b) Let M and m be the maximum and minimum respectively, of the function f. Show that for any $x \in [0, 1]$, $t \geqslant 0$ we have $e^{-t}m \leqslant u(x, t) \leqslant e^{-t}M$.

10 4. Find all functions h such that the function $u(x, y, t) = \frac{1}{t}h(\frac{x^2+y^2}{t})$ satisfies the heat equation $\partial_t u - \frac{1}{2}\triangle u = 0$. [Recall that in this case $\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. Note that h is a function of one variable.]

Please don't do the extra credit, until you have completed all other problems, *and* checked your work throughly. You will *not* be awarded partial credit on the extra credit problem.

5 5. *(Extra credit)* Let $\alpha \in \mathbb{R}$, and h be a function of one variable. Suppose $u(\vec{x}, t) = \frac{1}{t^\alpha}h(\frac{\|\vec{x}\|^2}{t})$ satisfies the heat equation $\partial_t u - \frac{1}{2}\triangle u = 0$ in $\mathbb{R}^n$. Find α. [Here $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and $\triangle u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$. Note that you aren't required to find h. Just α. You get half credit if you do this for $n = 3$.]