# Math 131 Midterm

Tue, October 23, 2007

Time: 75 mins
Total: 50 points

*This is a closed book test. Calculators can be kept handy for your 'moral comfort', but they will not be of any use at all. Please don't use cell phones or pagers. Good luck* ◡̈

[10] 1. Solve the PDE $u_x + u_y = u^3$, given $u(0, y) = f(y)$. Is the solution (from the previous part) defined on the entire $xy$-plane? Justify.

[5] 2. (a) Suppose $v$ satisfies the wave equation $v_{tt} - c^2 v_{xx} = 0$. If $v(x, 0) = v_t(x, 0) = 0$ whenever $|x| \leqslant 1$, then find the largest region in the $xt$-plane with $t \geqslant 0$, where you can guarantee that $v(x, t) = 0$.

[5] (b) Let $f$ be some function of $x$ and $t$. Suppose $u_1$ and $u_2$ are two solutions of the forced wave equation $u_{tt} - c^2 u_{xx} = f$, with initial data $u_1(x, 0) = \varphi_1(x)$, $\partial_t u_1(x, 0) = \psi_1(x)$, $u_2(x, 0) = \varphi_2(x)$, $\partial_t u_2(x, 0) = \psi_2(x)$ respectively. Suppose whenever $|x| \leqslant 1$, $\varphi_1(x) = \varphi_2(x)$ and $\psi_1(x) = \psi_2(x)$. Find the largest region in the $xt$-plane (with $t \geqslant 0$) where you can guarantee $u_1(x, t) = u_2(x, t)$. [HINT: This is part (b) of a question.]

[10] 3. Let $u$ satisfy the transport equation $\partial_t u + c\partial_x u = 0$, with initial data $u(x, 0) = f(x)$. Show that $\int_{-\infty}^{\infty} |u(x, t)|\, dx$ is constant in time. [As you know very well, the absolute value function is *not* differentiable. So please don't provide a proof that differentiates the absolute value function. You can assume that $\int_{-\infty}^{\infty} |f| < \infty$.]

[9] 4. (a) Let $a$ and $b$ be two functions such that $a(x, t) \geqslant 0$ for all $x, t$. Suppose $u$ satisfies $u_t + bu_x - au_{xx} < 0$ on the rectangle $[0, L] \times [0, T]$. Show that $u$ satisfies the strong maximum principle: Namely show that $u$ does not attain it's maximum on the interior, or top of the rectangle $[0, L] \times [0, T]$. [HINT: The argument is mostly the same as the proof we had in class. You need to make sure nothing goes wrong with our proof when the diffusion coefficient is not constant (but still positive), and you have to figure out what to do with the $bu_x$ term.]

[1] (b) What part of your proof in the previous subpart *fails* if you do not assume $a(x, t) \geqslant 0$.

[10] 5. Suppose $u$ satisfies the heat equation $\partial_t u - \frac{1}{2}\partial_{xx} u = 0$ on $\mathbb{R}$, with initial data $u(x, 0) = f(x)$ and vanishes at infinity (i.e. $\lim_{x\to\pm\infty} u(x, t) = 0$ for any $t$). Assume $f$ is such that $\int_{-\infty}^{\infty} |f| < \infty$. Show that $\lim_{t\to\infty} u(x, t) = 0$. [HINT: You can assume that $u$ is given by the explicit formula we had in class. Use this formula to show that for some suitable constant $c$, we have $|u(x, t)| < \frac{c}{\sqrt{t}}$ for any $t > 0$. As an (unrelated) remark about the physical significance, the conclusion of this problem is 'natural': If you supply a *finite* amount of heat to an *infinitely* long rod, eventually it should all dissipate and the temperature should become uniformly 0.]