Assignment 13: Assigned Wed 04/18. Due Wed 04/25

1. **Sec. 6.4.** 4, 12.

2. Let D be a disc with center 0 and radius 1. Let f be some function with $\int_{-\pi}^{\pi} f(\theta)^2\,d\theta < \infty$, and u be the solution of $-\triangle u = 0$ in D with $u = f$ on ∂D. Define g to be the indefinite integral

$$g(\theta) = \int \lim_{r\to 1^-} \partial_r u(r,\theta)\,d\theta + c,$$

where the constant of integration c is chosen such that $\int_{-\pi}^{\pi} g(\theta)\,d\theta = 0$. The function g is called the (periodic) *Hilbert Transform* of f.

   (a) Compute the (complex) Fourier coefficients of g in terms of those of f. [For this subpart, feel free to pass the appropriate derivatives/integrals/limits through an infinite sum. You'll get extra credit for rigorously justify all the operations you do.]

   (b) Guess a formula (and explain your guess) for a function K so that $g(\theta) = \int_{-\pi}^{\pi} f(\phi)K(\theta-\phi)\,d\phi$. [The reason I say guess is because you will have $\int_{-\pi}^{\pi} |K(\phi)|\,d\phi = +\infty$; consequently, the integral $\int_{-\pi}^{\pi} K(\theta-\phi)f(\phi)\,d\phi$ will be *undefined* in the usual Riemann (or even Lebesgue!) sense. If f is differentiable it turns out that the symmetric limit $\lim_{\varepsilon\to 0^+}\int_{|\theta-\phi|>\varepsilon} f(\phi)K(\theta-\phi)\,d\phi$ will always exist. Extending this to the situation where f is merely continuous (or just integrable!) but requires some non-trivial Harmonic analysis developed by Calderón and Zygmund. Wikipedia 'Hilbert Transform' to see some applications.]

   (c) If $\int_{-\pi}^{\pi} f = 0$, find a relationship between $\|f\|$ and $\|g\|$.

   (d) Using part (a), guess a formula for the (complex) Fourier coefficients of K. Verify your guess by computing explicitly

$$\frac{1}{2\pi}\lim_{\varepsilon\to 0^+}\int_{\substack{\theta\in[-\pi,\pi]\\ |\theta|>\varepsilon}} K(\theta)e^{in\theta}\,d\theta$$

3. Suppose $D \subseteq \mathbb{R}^2$ is a bounded domain completely contained inside a disk of radius R. Suppose τ is the solution of $-\triangle\tau = 1$ in D, and $\tau = 0$ on ∂D. What sign must τ have in D? Find a constant $c > 0$, which *only depends on* R such that $\tau(x) \leqslant c$ for all $x \in D$. [Hint: The maximum principle quickly implies that if $-\triangle u \geqslant 1$ in D and $u \geqslant \tau$ on ∂D, then $u \geqslant \tau$ inside D. Cleverly choose u. Unrelated trivia: If you start a continuous time random walker (Brownian motion) at the point $x \in D$, then average time it will take to exit D is exactly $2\tau(x)$.]

4. Let $D = [0,L]\times[0,L]$, and b be some (bounded) vector function. Suppose u is a solution of the PDE $-\triangle u + b\cdot\nabla u = b_1$, with $u = 0$ on ∂D. (Note: b_1 is the first component of the vector b.) Find a constant c which *only depends on* L such that $u(x) \leqslant c$ for all $x \in D$. [This is a short, but tricky, application of the maximum principle.]

5. **Sec. 7.1.** 5, 7

Assignment 14: Assigned Wed 04/25. Due Wed 05/02

1. **Sec. 7.1.** 6.

2. **Sec. 7.2.** 1.

3. **Sec. 7.3.** 1, 2

4. **Sec. 7.4.** 1, 3.

5. *(Hopf lemma revisited)* Here's a simpler way to do 6.4.12 than the online solution. (Consequently, you may *not* use the Hopf lemma for this proof.)

   (a) Given $0 < R_0 < R_1$, let $A(R_0,R_1)$ be the annulus $\{x\in\mathbb{R}^2 \mid R_0 < |x| < R_1\}$. Given two constants c_0 and c_1, find the solution to the PDE $-\triangle v = 0$ in $A(R_0,R_1)$, with $v = c_0$ on the inner boundary, and $v = c_1$ on the outer boundary.

   (b) If $c_0 < c_1$, verify that $\partial_r v(R_1,\theta) > 0$.

   (c) Let $B_R = \{x\in\mathbb{R}^2 \mid |x| < R\}$. Suppose u is some function such that $-\triangle u \leqslant 0$ in B_R, and u attains a maximum at some point $x_0 \in \partial B_R$. Suppose further $u(0) < u(x_0)$. Show that $\partial_r u(x_0) > 0$. [Hint: Observe first that for some R_0 small enough, $c_0 = \max_{|x|=R_0} < u(x_0)$. Let $c_1 = u(x_0)$ and use the maximum principle and previous subparts.]

   (d) *(Strong maximum principle)* Suppose $D \subseteq \mathbb{R}^2$ is some domain, and u is a *non-constant* function with $-\triangle u \leqslant 0$ in D. Show that u can not attain an interior maximum.

   (e) *(Hopf lemma)* Suppose $D \subseteq \mathbb{R}^2$ is a domain with a smooth boundary. Suppose u is a non-constant function satisfying $-\triangle u \leqslant 0$ in D, and is continuous up to the boundary of D. If u attains it's maximum at a point $x_0 \in \partial D$, show that $\frac{\partial u}{\partial \hat{n}} > 0$ at x_0, where $\hat{n}$ is the outward pointing unit normal vector.