Assignment 13: Assigned Wed 04/18. Due Wed 04/25

1. Sec. 6.4. 4, 12.

2. Let $D$ be a disk with center 0 and radius 1. Let $f$ be some function with $\int_{-\pi}^{\pi} f(\theta)^2 d\theta < \infty$, and $u$ be the solution of $-\Delta u = 0$ in $D$ with $u = f$ on $\partial D$. Define $g$ to be the indefinite integral

$$g(\theta) = \int \lim_{r \to 1^-} \partial_r u(r, \theta) d\theta + c,$$

where the constant of integration $c$ is chosen such that $\int_{-\pi}^{\pi} g(\theta) d\theta = 0$. The function $g$ is called the (periodic) Hilbert Transform of $f$.

(a) Compute the (complex) Fourier coefficients of $g$ in terms of those of $f$. [For this subpart, feel free to pass the appropriate derivatives/integrals/limits through an infinite sum. You'll get extra credit for rigorously justify all the operations you do.]

(b) Guess a formula (and explain your guess) for a function $K$ so that $g(\theta) = \int_{-\pi}^{\pi} f(\phi) K(\theta - \phi) d\phi$. [The reason I say guess is because you will have $\int_{-\pi}^{\pi} |K(\phi)| d\phi = +\infty$; consequently, the integral $\int_{-\pi}^{\pi} K(\theta - \phi) f(\phi) d\phi$ will be undefined in the usual Riemann (or even Lebesgue!) sense. If $f$ is differentiable it turns out that the symmetric limit $\lim_{\varepsilon \to 0^+} \int_{|\theta - \phi| > \varepsilon} f(\phi) K(\theta - \phi) d\phi$ will always exist. Extending this to the situation where $f$ is merely continuous (or just integrable!) but requires some non-trivial Harmonic analysis developed by Calderón and Zygmund. Wikipedia ‘Hilbert Transform’ to see some applications.]

(c) If $\int_{-\pi}^{\pi} f = 0$, find a relationship between $\|f\|$ and $\|g\|$.

(d) Using part (a), guess a formula for the (complex) Fourier coefficients of $K$.

Verify your guess by computing explicitly

$$\frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_{\theta \in [-\pi, \pi]} K(\theta) e^{i n \theta} d\theta$$

3. Suppose $D \subseteq \mathbb{R}^2$ is a bounded domain completely contained inside a disk of radius $R$. Suppose $\tau$ is the solution of $-\Delta \tau = 1$ in $D$, and $\tau = 0$ on $\partial D$. What sign must $\tau$ have in $D$? Find a constant $c > 0$, which only depends on $R$ such that $\tau(x) \leq c$ for all $x \in D$. [HINT: The maximum principle quickly implies that if $-\Delta u > 1$ in $D$ and $u \geq \tau$ on $\partial D$, then $u \geq \tau$ inside $D$.] Cleverly choose $u$. Unrelated trivia: If you start a continuous time random walker (Brownian motion) at the point $x \in D$, then average time it will take to exit $D$ is exactly $2\tau(x).$

4. Let $D = [0, L] \times [0, L]$, and $b$ be some (bounded) vector function. Suppose $u$ is a solution of the PDE $-\Delta u + b \cdot \nabla u = b_1$, with $u = 0$ on $\partial D$. [Note: $b_1$ is the first component of the vector $b$.] Find a constant $c$ which only depends on $L$ such that $u(x) \leq c$ for all $x \in D$. [This is a short, but tricky, application of the maximum principle.]

Assignment 14: Assigned Wed 04/25. Due Wed 05/02

1. Sec. 7.1. 6.

2. Sec. 7.2. 1.

3. Sec. 7.3. 1, 2

4. Sec. 7.4. 1, 3

5. (Hopf lemma revisited) Here’s a simpler way to do 6.4.12 than the online solution. (Consequently, you may not use the Hopf lemma for this proof.)

(a) Given $0 < R_0 < R_1$, let $A(R_0, R_1)$ be the annulus $\{x \in \mathbb{R}^2 \mid R_0 < |x| < R_1\}$. Given two constants $c_0$ and $c_1$, find the solution to the PDE $-\Delta v = 0$ in $A(R_0, R_1)$, with $v = c_0$ on the inner boundary, and $v = c_1$ on the outer boundary.

(b) If $c_0 < c_1$, verify that $\partial_r v(R_1, \theta) > 0$.

(c) Let $B_R = \{x \in \mathbb{R}^2 \mid |x| < R\}$. Suppose $u$ is some function such that $-\Delta u \not\equiv 0$ in $B_R$, and $u$ attains a maximum at some point $x_0 \in \partial B_R$. Suppose further $u(0) < u(x_0)$. Show that $\partial_r u(x_0) > 0$. [HINT: Observe first that for some $R_0$ small enough, $c_0 = \max_{|x|=R_0} u(x_0)$. Let $c_1 = u(x_0)$ and use the maximum principle and previous subparts.]

(d) (Strong maximum principle) Suppose $D \subseteq \mathbb{R}^2$ is some domain, and $u$ is a non-constant function with $-\Delta u \leq 0$ in $D$. Show that $u$ cannot attain an interior maximum.

(e) (Hopf lemma) Suppose $D \subseteq \mathbb{R}^2$ is a domain with a smooth boundary. Suppose $u$ is a non-constant function satisfying $-\Delta u \leq 0$ in $D$, and is continuous up to the boundary of $D$. If $u$ attains it’s maximum at a point $x_0 \in \partial D$, show that $\frac{\partial u}{\partial n} > 0$ at $x_0$, where $n$ is the outward pointing unit normal vector.