Assignment 10: Assigned Wed 03/28. Due Wed 04/04

1. **Sec. 5.3.** 2, 4.
2. **Sec. 5.4.** 1, 12, 13 [Assume that the Fourier series converges in L^2].
3. Let V be the set of all complex valued functions such that $\int_{-L}^{L} |f|^2 < \infty$, and $f(x+2L) = f(x)$. Define $\langle f, g\rangle = \int_{-L}^{L} f(x)\overline{g(x)}\,dx$.
   (a) If $f, g \in V$ show that $\langle i\partial_x f, g\rangle = \langle f, i\partial_x g\rangle$.[Trivia: Up to a constant, $i\partial_x$ is the momentum operator in Quantum mechanics.]
   (b) If $f, g \in V$, show that $\langle \partial_x^2 f, g\rangle = \langle f, \partial_x^2 g\rangle$.
   (c) Compute the eigenvalues and eigenfunctions of $i\partial_x$ with periodic boundary conditions as in the definition of V.
4. Using notation from the previous question, let T be any operator that satisfies the property $\langle Tf, g\rangle = \langle f, Tg\rangle$. When the functions are real valued, operators with this property are called *Symmetric*, as we saw in class. When the functions are complex valued, such operators are called *Hermitian*.
   (a) If for some $\lambda \in \mathbb{C}$, we have $Tf = \lambda f$ then show that $\lambda \in \mathbb{R}$.
   (b) If $\lambda \neq \mu \in \mathbb{C}$, and $f, g \in V$ are such that $Tf = \lambda f$ and $Tg = \mu g$. Show that $\langle f, g\rangle = 0$. [You'll need to use the previous subpart!]
5. Find a sequence of functions (f_n) such that $\int_{-\infty}^{\infty} |f_n(x)|^2\,dx < \infty$, $(f_n) \to 0$ uniformly on $(-\infty, \infty)$, however (f_n) does *not* converge to 0 in $L^2(-\infty, \infty)$. Why does this not contradict the result from class?

Assignment 11: Assigned Wed 04/04. Due Wed 04/11

1. **Sec. 5.4.** 8, 10. [#10 uses #9, but I did a version of #9 in class so didn't put it on this HW.]
2. Let f be a function such that $\int_0^L f^2 < \infty$, and let u be the solution to $u_t - \kappa u_{xx} = 0$ for $x \in (0, L)$, $t > 0$, with boundary conditions $u(0,t) = u(L,t) = 0$ and initial data $u(x, 0) = f(x)$.
   (a) For $t \geqslant 0$, let $B_n(t) = \frac{2}{L}\int_0^L u(x,t)\sin(\frac{n\pi}{L}x)\,dx$ be the Fourier Sine coefficients of u. For any $s \geqslant 0$, show that $\sum_1^\infty (n^s B_n)^2 < \infty$. [Hint: You know what $B_n(t)$ is explicitly as a function of $B_n(0)$ and t.]
   (b) Show that for any $t > 0$, u is infinitely differentiable. [Use without proof the Sobolev embedding theorems.]
   (c) Show further $\|u\|^2 \leqslant \exp(-\frac{2\pi^2}{L^2}\kappa t)\|f\|^2$. [Thus as $t \to \infty$, $u(\cdot, t)$ converges to 0 in $L^2(0, L)$ at an exponentially fast rate.]
   (d) Show that $\lim_{t\to\infty} \max_{x\in[0,L]} u(x,t) = 0$. [Thus as $t \to \infty$, $u(\cdot, t)$ converges to 0 uniformly.]
3. Let f be a (real valued), $2L$-periodic function such that $\int_{-L}^{L} f(x)^2\,dx < \infty$, $\int_{-L}^{L} f'(x)^2\,dx < \infty$ and $\int_{-L}^{L} f(x)\,dx = 0$. What is the minimum $\frac{\|f'\|^2}{\|f\|^2}$ can be? Prove it. Also find a function f which attains this minimum value. [Hint: Use the full Fourier series expansion for f and f'.]
4. Let f be a complex valued $2L$-periodic function, and $c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi}{L}x}\,dx$ be the n^{th} Complex Fourier coefficient of f. Let $S_N f = \sum_{-N}^{N} c_n f_n$ be the partial sums and $\sigma_N f = \frac{1}{N}\sum_0^{N-1} S_N f$.
   (a) Show that $\sigma_N f(x) = \int_{-L}^{L} K_N(x-y)f(y)\,dy$, where $K_N(z) = \frac{\sin\left(\frac{N}{2}\frac{\pi}{L}x\right)^2}{2NL\sin\left(\frac{1}{2}\frac{\pi}{L}x\right)^2}$
   (b) Show that there exists a constant $C > 0$ such that for all N, x, we have $K_N(x) \leqslant \frac{C}{N}\max\{N^2, \frac{1}{x^2}\}$.
   (c) For all N, show that $K_N \geqslant 0$, and $\int_{-L}^{L} K_N(x)\,dx = 1$.
   (d) For any $\varepsilon > 0$, show that $\lim_{N\to\infty}\int_{-L}^{-\varepsilon} K_N(x)\,dx + \int_{\varepsilon}^{L} K_N(x) = 0$.
   (e) If f is continuous at the point $x \in [-L, L]$, then show that $\lim_{N\to\infty}\sigma_N f(x) = f(x)$. [If you know uniform continuity, this proof will also show $\sigma_N f$ will converge to f uniformly.]

Assignment 12: Assigned Wed 04/11. Due Wed 04/18

1. **Sec. 6.1.** 6, 9, 11.
2. **Sec. 6.3.** 1, 4.
3. Suppose u is a function of n variables such that $-\triangle u = 0$. Suppose further, $u(x_1, \dots, x_n) = f(x_1^2 + \cdots + x_n^2)$, for some function f. Find f. [These are radial Harmonic functions.]
4. Let $P(r, \theta)$ be the Poisson kernel on a disk of radius a. For any $\varepsilon > 0$, show that $\lim_{r\to a^-}\left[\int_{-\pi}^{-\varepsilon} P(r,\theta)\,d\theta + \int_{\varepsilon}^{\pi} P(r,\theta)\right] = 0$.
5. Let D be a disk of radius a, and u be the solution of $-\triangle u = 0$ in D, with boundary condition $u(a, \theta) = f(\theta)$. Suppose $\int_{-\pi}^{\pi} |f(\theta)|^2\,d\theta < \infty$. Let $c_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(r,\theta)e^{-in\pi\theta}\,d\theta$ be the complex Fourier coefficients of $u(r, \cdot)$.
   (a) For any $s \geqslant 0$ and $r < a$, show that $\sum_{-\infty}^{\infty} |n^s c_n(r)| < \infty$. [As before, you know $c_n(r)$ explicitly in terms of $c_n(a)$.]
   (b) Show that for any $r < a$, u is infinitely differentiable.
   (c) Show that $\lim_{r\to a^-}\int_{-\pi}^{\pi} |u(r,\theta) - f(\theta)|^2\,d\theta = 0$. [This is one situation where the theorem allowing you to interchange the limit and integral *does not* apply. You'll have to do this out explicitly. Hint – Fourier series ..., but perhaps you guessed that already.]