Assignment 6: Assigned Wed 02/22. Due Wed 02/29

1. (a) Compute the solution of $\partial_t u - \frac{1}{2}\partial_{xx}^2 u = f$ given $u(x,0) = 0$, and $f(x,t) = 1$ if $|x| \leqslant 1$, and $f(x,t) = 0$ otherwise.

   (b) For $\alpha \geqslant 0$, compute $\lim_{t\to\infty} \frac{1}{t^\alpha} u(x,t)$.

2. Suppose we want to solve $\partial_t u - \kappa \partial_x^2 u = 0$ for $x \in (0,\infty)$, $t > 0$ with $u(x,0) = f(x)$ and Dirichlet boundary conditions $u(0,t) = 0$. Here's a trick:

   (a) Let $g(x) = f(x)$ for $x \geqslant 0$, and $g(x) = -f(-x)$ for $x < 0$ (this is called the *odd* extension of f). Let v be the solution of $\partial_t v - \kappa\partial_x^2 v = 0$ for $x \in \mathbb{R}$, $t > 0$ with initial data $v(x,0) = g(x)$. Show that $v(0,t) = 0$ for all $t > 0$. [You may, but need not, use the explicit solution formula. You may assume all functions decay at infinity.]

   (b) Use the explicit formula for v from the previous subpart to show that $u(x,t) = \int_0^\infty f(y)[G(x-y,t) - G(x+y,t)]\,dy$ is the desired solution. [You don't have to check u has the correct initial data yet; but you should justify why u satisfies the PDE, and boundary conditions.]

   (c) Find an explicit formula for the solution of the PDE $\partial_t u - \kappa\partial_x^2 u = g(x,t)$, for $x > 0, t > 0$, with Dirichlet boundary conditions $u(0,t) = 0$ and initial data $u(x,0) = f(x)$. [You may assume decay as $x \to +\infty$.]

3. Finish the ODE calculation from class: Suppose $\partial_t^2 S - AS = 0$, with $S(x,0) = 0$, and $\partial_t S(x,0) = x$. Then show directly $y(t) = \partial_t S(a,t) + S(b,t) + \int_0^t S(g(s), t-s)\,ds$ solves $\frac{d^2y}{dt^2} - Ay = g$ with $y(0) = a$ and $\frac{dy}{dt}(0) = b$.

4. **Sec. 3.4.** 3, 5.

5. (a) Use the same trick we used for the heat equation to find an explicit formula for the solution of the PDE $\partial_t^2 u - c^2\partial_x^2 u = 0$, for $x > 0, t > 0$, with Dirichlet boundary conditions $u(0,t) = 0$ and initial data $u(x,0) = \varphi(x)$, and $\partial_t u(x,0) = \psi(x)$.

   (b) For the previous subpart, sketch the domain of dependence of a point (x,t). [Do two cases: $x < ct$ and $x \geqslant ct$. Your pictures will be different!]

   (c) Appropriately modify the previous trick to find an explicit formula for the solution of the PDE $\partial_t^2 u - c^2\partial_x^2 u = g(x,t)$, for $x > 0, t > 0$, with Neumann boundary conditions $\partial_x u(0,t) = 0$ and initial data $u(x,0) = \varphi(x)$, and $\partial_t u(x,0) = \psi(x)$.

Assignment 7: Assigned Wed 02/29. Due Wed 03/07

1. Let f be a bounded function, and $u(x,t) = \int_{-\infty}^{\infty} f(y)G(x-y,t)\,dy$.

   (a) Show that $\lim_{t\to 0^+} u(x,t) = \frac{1}{2}(f(x^+) + f(x^-))$, where $f(x^\pm) = \lim_{y\to x\pm} f(y)$.

   (b) (Unrelated) If f is differentiable and f' is continuous at x, show that $\lim_{t\to 0^+} \partial_x u(x,t) = f'(x)$. [Hint: Express $\partial_x u$ in terms of f'.]

2. **Sec. 4.1.** 3, 4, 6.

3. **Sec. 4.2.** 2, 4.

4. Let u be a solution of $\partial_t u - \partial_x^2 u = 0$ for $x \in (0,1)$, with initial data $u(x,0) = 1$, and boundary conditions $u(0,t) = u(1,t) = 0$ for $t > 0$. Show that

$$\max_{x\in(0,1)} u(x,t) \geqslant e^{-\pi^2 t}.$$

Assignment 8: Assigned Wed 03/07. Due Wed 03/21

1. **Sec. 5.1.** 2, 5

2. This problem extends the symmetry and orthogonality lemmas to higher dimensions. Let D be a bounded region in $\mathbb{R}^3$ (or in $\mathbb{R}^2$).

   (a) Let u, v be two functions. Show that $\int_D u\triangle v = \int_{\partial D} u\frac{\partial v}{\partial \hat{n}} - \int_D (\nabla u)\cdot(\nabla v)$. [Recall, by $\int_D f$, I mean the volume integral $\iiint_D f(x,y,z)\,dV$. Similarly by $\int_{\partial D} f$, I mean the surface integral $\iint_D f(x,y,z)\,dS$.]

   (b) *(Positivity)* Suppose $-\triangle u = \lambda u$ in D, with the Dirichlet boundary condition $u = 0$ on the boundary of D. Show that $\lambda = \left(\int_D |\nabla u|^2\right)/\left(\int_D u^2\right)$, and hence $\lambda \geqslant 0$.

   (c) *(Symmetry)* Suppose u and v satisfy the Dirichlet boundary conditions $u = v = 0$ on the boundary of D. Show that $\int_D (-\triangle u)v = \int_D u(-\triangle v)$.

   (d) *(Orthogonality)* Suppose $-\triangle u = \lambda u$, $-\triangle v = \mu v$, $\lambda \neq \mu$, and u, v satisfy the Dirichlet boundary conditions $u = v = 0$ on the boundary of D. Show that $\int_D uv = 0$.

   (e) Do the previous subparts work if we replace the Dirichlet boundary conditions with Neumann? Explain.

Assignment 9: Assigned Wed 03/21. Due Wed 03/28

1. Let f be an even $2L$ periodic function.

   (a) If f is differentiable show that $f'(0) = f'(L) = 0$.

   (b) Show that the full Fourier series for f on the interval $[-L, L]$ is the same as the Fourier Cosine series for f on the interval $[0, L]$.

   (c) If instead f is odd, what is the analogue of the previous two subparts?

2. Let $f(x) = \sin(x)$ on the interval $(0,\pi)$.

   (a) Compute the Fourier Sine and the Fourier Cosine series for f. [Note that f only satisfies the boundary conditions required for the Fourier Sine series, and not those required for the Fourier Cosine series. It turns out that the Fourier Cosine series will converge to f at every point *except* possibly at the two boundary points!]

   (b) Also compute the Full Fourier series and Complex Fourier series for f on the interval $(-\pi,\pi)$.

3. If f is a real valued function, and a_n be the n^{th} complex Fourier coefficient of f. Show that $a_{-n} = \overline{a_n}$.

4. Suppose $f = \frac{A_0}{2} + \sum_1^\infty A_n\cos(\frac{n\pi}{L}x) + B_n\sin(\frac{n\pi}{L}x)$ and $f = \sum_{-\infty}^\infty c_n\exp(i\frac{n\pi}{L}x)$. Find a relation between the c_n's, A_n's and B_n's.