Assignment 4: Assigned Wed 02/08. Due Wed 02/15

1. **Sec. 2.3.** 3, 5, 6, 7.
2. (a) Let $L, T > 0$, and a, b be two continuous functions such that $a(x,t) \geqslant 0$ (with no assumption on b). Suppose u is a function such that $u_t + bu_x - au_{xx} < 0$ on the rectangle $R = (0,L) \times (0,T)$, and u is continuous on the rectangle $[0,L] \times [0,T]$. Show that u attains it's maximum *only* on the sides or bottom of this rectangle.

   (b) Suppose instead $u_t + bu_x - au_{xx} \leqslant 0$ above. Show that u attains it's maximum on the sides or bottom of the above rectangle. [Your proof will (probably) not rule out the possibility that u also attains it's maximum on the interior of R. Ruling out interior maxima is the content of the strong maximum principle, which is a lot harder to prove.]

   (c) Show that solutions to the PDE $u_t + bu_x - au_{xx} = f$, with initial data $u(x,0) = \varphi(x)$, and Dirichlet boundary conditions $u(0,t) = g_1(t)$ and $u(L,t) = g_2(t)$ are unique. That is, if u_1 and u_2 are two solutions to the above PDE, with the same initial data and boundary conditions, show that $u_1 = u_2$. [If a is not constant in x, then you won't (easily) be able to prove uniqueness to this PDE using energy methods.]
3. Suppose now $D \subseteq \mathbb{R}^n$ is a bounded region. Let $\bar{D} = D \cup \partial D$. For $i \in \{1, \dots, n\}$, let a_i, b_i be two continuous functions with $a_i \geqslant 0$, and no assumption on b. Suppose u satisfies the partial differential inequality

$$\partial_t u + \sum\nolimits_{i=1}^n b_i \partial_i u - \sum\nolimits_{i=1}^n a_i \partial_i^2 u \leqslant 0 \tag{1}$$

   in the region D. Show that the maximum of u on the region $\bar{D} \times [0,T]$ is attained on the sides or bottom (i.e. is attained either when $x \in \partial D$, or when $t = 0$). [As I mentioned in class, the maximum principle is true in greater generality. Namely, if you replace the second order terms with the operator $\sum_{i,j=1}^n a_{i,j} \partial_i \partial_j u$ for some symmetric, positive definite matrix $A = (a_{i,j})$. The only extra ingredient you need to carry out the proof in this case is the fact that at a local maximum, $\sum_{i,j=1}^n a_{i,j} \partial_i \partial_j u \leqslant 0$. This follows quickly from the spectral theorem if you've seen it. If not, no reason to worry: most interesting problems at this level involve equations in one dimension, or equations in the simplified form (1), with a_i all constant and equal.]
4. Let $D \subseteq \mathbb{R}^3$ be a bounded region. The evolution of the velocity field of an ideal fluid in the region D is given by the Euler equations:

$$\partial_t u + (u \cdot \nabla) u + \nabla p = 0 \qquad \text{and} \qquad \nabla \cdot u = 0$$

   Physically, u is the velocity field (an 3-dimensional vector), and p is the pressure (a scalar). The boundary conditions usually imposed are that $u \cdot \hat{n} = 0$ on the boundary of D. (Physically, this means that the fluid does not flow in or out of the region D.) Let $E(t) = \frac{1}{2} \int_D |u(x,t)|^2 \, dV$ (physically, this represents the kinetic energy of the fluid). Show that E is constant in time. [Note if the i^{th} component of the vector function u is u_i, then $(u \cdot \nabla)u$ is defined to be the vector who's i^{th} component is $\sum_j u_j \partial_j u_i$. Here $\partial_i = \frac{\partial}{\partial x_i}$ denotes the partial derivative with respect to the i^{th} coordinate.]

Assignment 5: Assigned Wed 02/15. Due Wed 02/22

1. **Sec. 2.4.** 6, 18.
2. Solve $u_t - \frac{1}{2} u_{xx} = 0$ on the line, with initial data $u(x,0) = |x|$. Sketch profiles of u for $t = \frac{1}{2}$, $t = 1$, $t = 10$. [This problem will show you how the corners of the initial data get smoothed out.]
3. Check that the heat equation has an infinite speed of propagation, in the following sense: If for any $\delta > 0$, we define $f(x) = 1$ when $|x| < \delta$ and $f(x) = 0$ otherwise. Let $u(x,t)$ solve $\partial_t u - \frac{1}{2} \partial_{xx}^2 u = 0$ for all $x \in \mathbb{R}$, $t > 0$ with initial data $u(x,0) = f(x)$. Then show that for any $t > 0$, $u(x,t) \neq 0$ for all x. [Thus the small heat source centered at 0 is *immediately* felt at all points x.]
4. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, define $G(x,t) = (2\pi t)^{-n/2} e^{-|x|^2/2t}$.

   (a) Show that $\partial_t G = \frac{1}{2} \triangle G$ for any $t > 0$.

   (b) Show that $\int_{\mathbb{R}^n} G(x,t) \, dx = 1$ for any $t > 0$. [By $\int_{\mathbb{R}^n} G(x,t) \, dx$, I mean the iterated integral $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G(x_1, \dots, x_n, t) \, dx_1 \cdots dx_n$.]

   (c) Write down a formula for a solution to the heat equation $\partial_t u - \frac{1}{2} \triangle u = 0$, for $x \in \mathbb{R}^n, t > 0$ with initial data $u(x,0) = f(x)$. Verify your formula solves the equation for $t > 0$. (Verifying that it has the correct initial data is harder, and will be handled later on.)