Assignment 4: Assigned Wed 02/08. Due Wed 02/15

1. Sec. 2.3. 3, 5, 6, 7.
2. (a) Let $L, T > 0$, and $a, b$ be two continuous functions such that $a(x, t) \geq 0$ (with no assumption on $b$). Suppose $u$ is a function such that $u_t + bu_x - au_{xx} < 0$ on the rectangle $R = (0, L) \times (0, T)$, and $u$ is continuous on the rectangle $[0, L] \times [0, T]$. Show that $u$ attains its maximum only on the sides or bottom of this rectangle.

(b) Suppose instead $u_t + bu_x - au_{xx} \leq 0$ above. Show that $u$ attains its maximum on the sides or bottom of the above rectangle. [Your proof will (probably) not rule out the possibility that $u$ also attains its maximum on the interior of $R$. Ruling out interior maxima is the content of the strong maximum principle, which is a lot harder to prove.]

(c) Show that solutions to the PDE $u_t + bu_x - au_{xx} = f$, with initial data $u(x, 0) = \varphi(x)$, and Dirichlet boundary conditions $u(0, t) = g_1(t)$ and $u(L, t) = g_2(t)$ are unique. That is, if $u_1$ and $u_2$ are two solutions to the above PDE, with the same initial data and boundary conditions, show that $u_1 = u_2$. [If $u$ is not constant in $x$, then you won’t (easily) be able to prove uniqueness to this PDE using energy methods.]

3. Suppose now $D \subseteq \mathbb{R}^n$ is a bounded region. Let $\bar{D} = D \cup \partial D$. For $i \in \{1, \ldots, n\}$, let $a_i, b_i$ be two continuous functions with $a_i \geq 0$, and no assumption on $b$. Suppose $u$ satisfies the partial differential inequality

$$
\partial_t u + \sum_{i=1}^n b_i \partial_i u - \sum_{i=1}^n a_i \partial_i^2 u \leq 0
$$

in the region $D$. Show that the maximum of $u$ on the region $\bar{D} \times [0, T]$ is attained on the sides or bottom (i.e. is attained either when $x \in \partial D$, or when $t = 0$). [As I mentioned in class, the maximum principle is true in greater generality. Namely, if you replace the second order terms with the operator $\sum_{i,j=1}^n a_{i,j} \partial_i \partial_j u$ for some symmetric, positive definite matrix $A = (a_{i,j})$. The only extra ingredient you need to carry out the proof in this case is the fact that at a local maximum, $\sum_{i,j=1}^n a_{i,j} \partial_i \partial_j u \leq 0$. This follows quickly from the spectral theorem if you’ve seen it.]

4. Let $D \subseteq \mathbb{R}^3$ be a bounded region. The evolution of the velocity field of an ideal fluid in the region $D$ is given by the Euler equations:

$$
\partial_t u + (u \cdot \nabla) u + \nabla p = 0 \quad \text{and} \quad \nabla \cdot u = 0
$$

Physically, $u$ is the velocity field (an 3-dimensional vector), and $p$ is the pressure (a scalar). The boundary conditions usually imposed are that $u \cdot \mathbf{n} = 0$ on the boundary of $D$. (Physically, this means that the fluid does not flow in or out of the region $D$.) Let $E(t) = \frac{1}{2} \int_D |u(x, t)|^2 dV$ (physically, this represents the kinetic energy of the fluid). Show that $E$ is constant in time. [Note if the $i$th component of the vector function $u$ is $u_i$, then $(u \cdot \nabla) u$ is defined to be the vector whose $i$th component is $\sum_{j} u_j \partial_j u_i$. Here $\partial_i = \frac{\partial}{\partial x_i}$ denotes the partial derivative with respect to the $i$th coordinate.]

Assignment 5: Assigned Wed 02/15. Due Wed 02/22

1. Sec. 2.4. 6, 18.
2. Solve $u_t - \frac{1}{2} u_{xx} = 0$ on the line, with initial data $u(x, 0) = |x|$. Sketch profiles of $u$ for $t = \frac{1}{2}$, $t = 1$, $t = 10$. [This problem will show you how the corners of the initial data get smoothed out.]

3. Check that the heat equation has an infinite speed of propagation, in the following sense: If for any $\delta > 0$, we define $f(x) = 1$ when $|x| < \delta$ and $f(x) = 0$ otherwise. Let $u(x, t)$ solve $\partial_t u - \frac{1}{2} \partial_x^2 u = 0$ for all $x \in \mathbb{R}, t > 0$ with initial data $u(x, 0) = f(x)$. Then show that for any $t > 0, u(x, t) \neq 0$ for all $x$. [Thus the small heat source centered at 0 is immediately felt at all points $x.$]

4. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define $G(x, t) = (2\pi t)^{-n/2} e^{-|x|^2/2t}$. Show that $\partial_t G = \frac{1}{2} \Delta G$ for any $t > 0$. [By the iterated integral $\int_{\mathbb{R}^n} G(x, t) dx, I mean the iterated integral $\int_{\mathbb{R}^n} G(x_1, \ldots, x_n, t) dx_1 \cdots dx_n.$]

(c) Write down a formula for a solution to the heat equation $\partial_t u - \frac{1}{2} \Delta u = 0$, for $x \in \mathbb{R}^n, t > 0$ with initial data $u(x, 0) = f(x)$. Verify your formula solves the equation for $t > 0$. (Verifying that it has the correct initial data is harder, and will be handled later on.)