Math 372: PDE Homework.

The problem numbers refer to problems from your text book (second edition). I will often assign problems which are not in the text book. *Keep in mind that there is a firm 'no late homework' policy.*

Assignment 1: Assigned Wed 01/18. Due Wed 01/25

1. **Sec. 1.1.** 3, 4
2. **Sec. 1.2.** 2, 3, 11, 12 [You don't need the 'coordinate method' to do 11, as the hint suggests. It can be done directly using the method of characteristics.]
3. Find the general solutions of the PDEs
   (a) $(1+x^2)\partial_x u + \partial_y u = yu^2$. (b) $(1-xy)\partial_x u + \partial_y u + \partial_z u = (y-z)u$.

Assignment 2: Assigned Wed 01/25. Due Wed 02/01

1. **Sec. 1.3.** 1, 2, 6, 10
2. Newtons law of cooling says that a body loses heat to it's surroundings at a rate proportional to the temperature difference. Consider a thin (1D) wire immersed in a medium of constant temperature θ_0, which exchanges heat with the surroundings according to Newtons law. Find a PDE satisfied by the temperature in the wire.
3. Let $\rho(x,y,t)$ be the density of a fluid at time t and position x, y. Let $u(x,y,t)$ be the instantaneous velocity of the fluid at time t and position x, y, and derive a PDE satisfied by ρ. [Hint: Assume that mass is conserved, and compute the rate of change of mass in some region D.]
4. **Sec. 1.4.** 3, 4, 6

Assignment 3: Assigned Wed 02/01. Due Wed 02/08

1. **Sec. 2.1.** 5, 10, 11 [See problem 9 for a hint on 10.]
2. **Sec. 2.2.** 2, 3.
3. (a) Suppose u satisfies $u_{tt} - c^2 u_{xx} = 0$ on the interval (a,b), for $t > 0$. Under what boundary conditions on u is energy is conserved? Prove it, and provide some physical explanation.

   (b) Let D be some region in (2 or 3) dimensional space. Suppose u satisfies $u_{tt} - c^2 \triangle u = 0$ in the region D. Define the energy to be the area/volume integral $E = \int_D (u_t^2 + c^2 |\nabla u|^2)\, dv$. Under what boundary conditions on u is energy conserved? Prove it. [Hint: From the previous part you should be able to guess the boundary conditions. For the proof, it requires replacing 'integration by parts' from your previous proof by a clever application of the divergence theorem.]
4. Let D be a region in $\mathbb{R}^3$, and $c, r > 0$ be constants, and a, f be functions depending only on the spatial variables x_1, x_2 and x_3. Show that solutions to the PDE
$$\partial_t^2 u - c^2 \triangle u + au + r\partial_t u = f$$
with Dirichlet boundary conditions
$$u = 0 \text{ on } \partial D$$
and initial data
$$u(x,0) = \varphi(x) \quad \& \quad \partial_t u(x,0) = \psi(x)$$
are unique. That is, if u_1 and u_2 are two solutions to the above PDE, with the same boundary conditions and initial data, show that they are equal. [Hint: Suppose u_1 and u_2 are two solutions. Set $v = u_1 - u_2$. Now try and cook up some 'energy' which will help you show v is 0. Note, the energy you cook up (if you do it right) won't be conserved! It will however decrease with time.]
5. Solitary waves (or solitons) are waves that travel great distances without changing shape. Tsunami's are one example. Scientific study began with Scott Russell in 1834, who followed such a wave in a channel on horseback, and was fascinated by it's rapid pace and unchanging shape. In 1895, Kortweg and De Vries showed that the evolution of the profile is governed by the equation
$$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0.$$
For this question, suppose u is a solution to the above equation for $x \in \mathbb{R}$, $t > 0$. Suppose further that u and all derivatives (including higher order derivatives) of u decay to 0 as $x \to \pm\infty$.
   (a) Let $p = \int_{-\infty}^{\infty} u(x,t)\, dx$. Show that p is constant in time. [Physically, p is the momentum of the wave.]
   (b) Let $E = \int_{-\infty}^{\infty} u(x,t)^2\, dx$. Show that E is constant in time. [Physically, E is the energy of the wave.]
   (c) It turns out that the KdV equation has *infinitely many* conserved quantities. The energy and momentum above are the *only two* which have any physical meaning. Can you find a non-trivial conserved quantity that's not a linear combination of p and E?